\( N \)-particle generalization of the Veneziano amplitude.

Search for a planar dual amplitude \( B_N \) in terms of all the planar energies
\[ \Delta_{ij} = (p_i + p_{i+1} + \ldots + p_j)^2. \]

There are \( \frac{1}{2} N(N-3) \) planar energies:

**Proof:** List them — e.g., \( N=6 \) case:

\[
\begin{align*}
\Delta_{12} & \quad \Delta_{23} & \quad \Delta_{34} \\
\Delta_{14} & \quad \Delta_{25} & \quad \Delta_{36} & \quad \Delta_{45} & \quad \Delta_{56} & \quad \Delta_{61} \\
\text{Vanishes} & \quad \text{is just } m^2 & \quad \text{by sum cons.} & \quad \text{by sum cons.}
\end{align*}
\]

\[ \Delta_{16} = (p_3 + p_4 + \ldots + p_5 + p_6)^2 \]
\[ \Delta_{15} = (p_3 + p_4 + p_5)^2 \]
\[ \Delta_{13} = (-p_6)^2 = m^2. \]

\[ \vdots \]
\[ \text{\# of planar energies} = (N-3) + \sum_{n=0}^{N-2} (N-3-n) = \frac{1}{2} N(N-2) \]

The full amplitude is then constructed by adding up all inequivalent planar orderings of the planar dual planar amplitudes.

There are \( \frac{1}{2} (N-1)! \) such orderings.

**Proof:** Number of cyclically inequivalent diagrams
\[ = \text{num. of unique ways to build a ring out of } N \text{ vertices}. \]

\[ = \frac{1}{2} (N-1)(N-2) \ldots = \frac{1}{2} (N-1)! \]

Building in reverse yields the same seq.
Constructing planar dual amplitude

As in the $N=5$ case, we try:

$$B_N(s_{12},...) = \int_0^1 dx_{12}... x_{12}^{-\alpha(s_{12})}... f(x_{12},...)$$

Should prevent overlapping divergences.

Write down the $\frac{1}{2}N(N-3)$ constraint equations:

$$x_{ij} = 1 - \prod_{(kl)} y_{kl} = 1 - \prod_{k=1}^{N-1} y_{ik} \prod_{m=i+1}^{N} x_{mn}$$  

over all overlapping variables

Looks impossible to solve, but it in fact a solution can be found.

There are $N-3$ redundant equations $\Rightarrow$ our integration variables $\Rightarrow$ Take them to be the variables in the first column of our list

Define: $y_2 = x_{12}$ This is our choice - we picked the $N-3$ redundant variables.

$y_3 = x_{13}$ $\rightarrow$ This corresponds to the multi-peripheral configuration:

$y_{N-2} = x_{1,N-2}$

Then, upon solving the constraint equations, find all other variables $x_{mn}$ are related to the $y_i$'s by:

$$x_{mn} = \frac{(1-y_m)(1-y_{m+1})...y_{n-1})(1-y_{m+1}...y_n)}{(1-y_m)...y_n)(1-y_{m+1}...y_n)} = \frac{(1-\prod_{i=m}^{n-1} y_i)(1-\prod_{i=m+1}^{n} y_i)}{(1-\prod_{i=m}^{n} y_i)(1-\prod_{i=m+1}^{n-1} y_i)}$$

with $y_1 = y_{N-1} = 0$ (they don't exist)

The Jacobian (without proof):

$$\left|\text{det} \left( \frac{\partial C_i}{\partial x_j} \right) \right| = \prod_{j=2}^{N-3} (1-y_j y_{j+1})^{-1}$$
So, the integrations run over the \( y \) variables

\[
B_N = \int_0^1 dy_2 \ldots dy_{N-2} \prod_{(m,n)} x_{mn}(y)^{-1-\alpha(s_{mn})} \left[ \frac{1}{\text{det} \left( \frac{\partial C}{\partial x_j} \right)} \right] \\
= \int_0^1 dy_2 \ldots dy_{N-2} \prod_{(m,n)} x_{mn}(y)^{-1-\alpha(s_{mn})} \prod_{j=2}^{N-3} \left( 1 - y_j \right)^{-1},
\]

where \( x_{1N} = y_N \) (convenient to pull out)

\[
B_N = \int_0^1 dy_2 \ldots dy_{N-2} \prod_{i=2}^{N-2} y_i^{1-\alpha(s_{i,i})} \prod_{m=2}^{N-2} \prod_{n=m+1}^{N-1} x_{mn}(y)^{-1-\alpha(s_{mn})} \prod_{j=2}^{N-3} \left( 1 - y_j \right)^{-1} 
\]

*All channel variables except for those from hat column.*

\[
y_2 \\
y_3 \quad x_2 \quad x_3 \\
y_4 \quad x_2 \quad x_3 \quad x_4 \\
\vdots \\
y_{N-2} \quad x_2 \quad x_3 \quad \ldots \quad x_{N-3} \quad x_{N-2} \\
x_2 \quad x_3 \quad \ldots \quad x_{N-3} \quad x_{N-2} \quad y_{N-2} \quad y_{N-1}
\]

Multi-peripheral configuration