Factorization of $N$-particle Veneziano amplitude

$$B_N = \int_0^1 dy_2 \ldots dy_{N-2} \prod_{i=2}^{N-2} y_i^{1-a(s_{i,i+1})} \prod_{m=2}^{N-2} \prod_{n=m+1}^{N-1} x_{mn}(y) \frac{1}{1-y_i y_{i+1}}$$

Suppose we take $S_{1k}$ such that $\alpha(s_{1k}) \to 0$.
Then: $y_{1k}$ integration diverges in the lower limit,

$$\Rightarrow \text{expand in plus det: } \frac{1}{y_{1k}^2 + \alpha(s_{1k})} = -\frac{1}{\alpha(s_{1k})} \delta(y_{1k}) + \ldots$$

$\text{pole in the } k\text{th internal line}$

Then, write

$$\prod_{i=2}^{N-2} y_i^{1-a(s_{i,i+1})} \rightarrow -\frac{1}{\alpha(s_{1k})} \delta(y_{1k}) \prod_{i=2}^{k-1} y_i^{1-a(s_{i,i+1})} \prod_{i=k+1}^{N-2} y_i^{1-a(s_{i,i+1})} + \ldots$$

All $(mn)$ channel variables overlapping with $1k$: $x_{mn} \rightarrow 1$.

$$\prod_{m=2}^{N-2} \prod_{n=m+1}^{N-1} x_{mn} \rightarrow \prod_{m=2}^{k-1} \prod_{n=m+1}^{N-1} x_{mn} \prod_{m=k+1}^{N-2} \prod_{n=m+1}^{N-1} x_{mn}$$

See: channel variables k-1:

\begin{overlapping variables

and finally,

$$\prod_{j=2}^{k-1} (1-y_j y_{j+1})^{-1} \rightarrow \prod_{j=2}^{k-2} (1-y_j y_{j+1})^{-1} \prod_{j=k+1}^{N-3} (1-y_j y_{j+1})^{-1}$$

So that $B_N$ factorizes at the pole (integrate over $y_{1k}$)

$$B_N = -\frac{1}{\alpha(s_{1k})} B_{k+1}(s_{1k}) B_N(s_{1k}) + \text{reg},$$

residue $\sim P_{\text{ex}}(\theta_k) (\text{spin-0 exc.})$

(Bootstrap condition)
-hard to show full factorization for higher poles \( \rightarrow \) operator formalism.

Possible to show residue at higher poles have the correct polynomial order:

Suppose \( S_{nk} \) is such that \( \alpha(S_{nk}) \rightarrow l \geq 0 \) (integer).

Isolate \( y_k \) integration variable

\[
B_N = \int_0^1 dy_k \ y_k^{1-\alpha(S_{nk})} \left[ \text{stuff} = F(y_k) \right]
\]

Series expand this around \( y_k = 0 \)
(Same as isolating poles in Bethe function)

\[
= \int_0^1 dy_k \ \frac{1}{y_k^{1+\alpha(S_{nk})-1}} \left[ \cdots + \frac{1}{l!} \frac{d^l}{dy_k^l} F(y_k) \right]_{y_k = 0} y_k^l + \cdots
\]

\[
= \int_0^1 dy \ \frac{1}{y^{1+\alpha(S_{nk})-1}} \left[ \frac{1}{l!} \frac{d^l}{dy^l} F(y) \right]_{y = 0} \frac{1}{\alpha(S)-l} \delta(y) + \frac{1}{\alpha(S)-l} \frac{d}{dy} \chi_{mn}(y)^{-1+\alpha(S_{mn})}
\]

Taking \( l \) derivatives will bring down at most \( l \) powers of \(-1+\alpha(S_{nk})\) dual variables.

\[
= \frac{-1}{\alpha(S)-l} \chi_{\text{polyn}(\text{in } S_{nk} \text{ of degree at most } l)}
\]

\[\Rightarrow\text{ no ancestors.}\]