On-shell conditions

Recall operator expression for scattering amplitude in multiperipheral configuration:

\[ B_N = \langle 0; p_1 | \hat{V} \hat{D} \hat{V} \cdots \hat{\sum}_{r_0} (r_0 - \omega_0) \frac{1}{L_0 - \omega_0 + r_0} \hat{V} \hat{D} \hat{V} \cdots | 0; p_2 \rangle \]

To factorize, we write one of the propagators as:

\[ \hat{S}(s) = \int_{0}^{1} dy \ (1-y)^{-4+\alpha_0} y^{-1-\alpha(s)} \sum_{\alpha} (-\alpha_0^\alpha \cdot \alpha_0) \]

\[ = \sum_{r_0=0}^{\infty} \left( \frac{r_0 - \omega_0}{r_0} \right) \frac{1}{\omega_0 + \sum_{\alpha} (-\alpha_0^\alpha \cdot \alpha_0) + r_0} \]

\[ \text{can write in terms of } l_0 = -\alpha_0^2 + \sum_{\alpha} (-\alpha_0^\alpha \cdot \alpha_0) \]

Write \( \alpha(s) = \alpha_0^2 + \alpha_0 \)

\[ \sum_{\alpha} (-\alpha_0^\alpha \cdot \alpha_0) = L_0 + \alpha_0^2 \]

will give \((p_3 \cdots + p_2)^2 = s_{23}\)

\[ \hat{S}(s) = \sum_{r_0=0}^{\infty} \left( \frac{r_0 - \omega_0}{r_0} \right) \frac{1}{L_0 - \omega_0 + r_0} \hat{S} \text{ (no longer explicitly dependent on } s \text{)} \]

so that

\[ B_N = \langle 0; p_1 | \hat{V} \hat{D} \hat{V} \cdots \sum_{r_0} (r_0 - \omega_0) \frac{1}{L_0 - \omega_0 - r_0} \hat{V} \hat{D} \hat{V} \cdots | 0; p_2 \rangle \]

Now, we will get a pole (correspondingly, a particle) when this \( l_0 \) gives an integer equal to \( \alpha_0 - r_0 \) upon acting on everything to the right (or left):

pole condition:

\[ l_0 \left[ \hat{V} \hat{D} \hat{V} \cdots | 0; p_2 \rangle \right] = (\alpha_0 - r_0) \left[ \hat{V} \hat{D} \hat{V} \cdots | 0; p_2 \rangle \right] \]

"tree"  "tree"
Since $[\hat{L}_0, \hat{D}] = 0$, it proves convenient to define $\text{tree}\rangle$ with an extra factor of $\hat{D}$ at end:

$$\text{tree}\rangle = \hat{D} \hat{\nu}(p_2) \hat{D} \hat{\nu}(p_{i-1}) \cdots |0; p_1\rangle$$

On-shell condition:

$$\hat{\Sigma}_0 |\text{tree}\rangle = (\alpha_0 - r_0) |\text{tree}\rangle$$

or

$$\left(\hat{\Sigma}_0 - \alpha_0 + r_0\right) |\text{tree}\rangle = 0$$

Furthermore, since $\hat{L}_n \sim \sum_{n=0}^{\infty} a_n a_{-m+n}$ for $n > 0$, $\hat{L}_n$ annihilates the physical ground state:

$$\hat{L}_n |0; p\rangle = 0 \text{ for } n > 0$$
On shell gauge conditions

For unit intercept, $\alpha_0 = 1$, there exists a remarkable set of on-shell gauge conditions satisfied by tree amplitudes (discovered by Virasoro):

First, the on-shell condition becomes ($\alpha_0 = 1 \Rightarrow \tau_0 = 0$)

$$(\hat{I}_0 - 1) |\text{tree}\rangle = 0.$$ 

Now derive two identities:

1. $$(\hat{I}_0 + n - 1 - \hat{n}) \hat{V}(p) = \hat{V}(p) (\hat{I}_0 + n - 1 - \hat{n}) + [I_0, \hat{V}(p)] - [\hat{n}, \hat{V}(p)]$$

   $$= \hat{V}(p) (\hat{I}_0 + n - 1 - \hat{n})$$

   $$z = 1 \quad \Rightarrow \quad + z^0 (z \frac{\partial}{\partial z} + (n+1)\alpha_0) \hat{V}(p) - z^n (z \frac{\partial}{\partial z} + (n+1)\alpha_0) \hat{V}(p)$$

   $$= \hat{V}(p) (\hat{I}_0 + n - 1 - \hat{n}) - n \alpha_0 \hat{V}(p).$$

   $$\Rightarrow \text{Removes } \alpha_0 \text{ dependence}$$

   $\hat{V}(p)$ must have unit conformal weight.

   $\alpha_0 = 1$ is crucial!

2. $$(\hat{I}_0 - 1 - \hat{n}) \frac{1}{\hat{I}_0 - 1} = 1 - \hat{n} \frac{1}{\hat{I}_0 - 1}$$

   For $\alpha_0 = 1$, this simplifies to

   $$= 1 + \hat{n} \sum_{m=0}^{\infty} \hat{I}_0^m$$

   $$= 1 + \sum_{m=0}^{\infty} (\hat{I}_0 + n)^m \hat{n}$$

   $$= 1 - \frac{1}{\hat{I}_0 + n - 1} \hat{n}$$

   $$= \frac{1}{\hat{I}_0 + n - 1} (\hat{I}_0 + n - 1 - \hat{n})$$

   $\Rightarrow \text{Removes } \hat{n}$$

Now consider $$(\hat{I}_0 - 1 - \hat{n}) |\text{tree}\rangle = (\hat{I}_0 - 1 - \hat{n}) \delta \psi \delta \ldots |0; p\rangle$$
\( (L_0 - 1 - \hat{1}_n)_{\text{tree}} = (L_0 - 1 - \hat{1}_n) \hat{D} \hat{D} \hat{D} \cdots |0; p_1 \rangle \)

Note: \( \hat{D} = \sum_{\alpha_0=0}^{\infty} \left( \frac{r_{\alpha_0} - \alpha_0}{r_{\alpha_0}} \right) \frac{1}{L_0 - \alpha_0 + r_{\alpha_0}} = \frac{1}{L_0 - 1} \) for \( \alpha_0 = 1 \)

Now push \((L_0 - 1 - \hat{1}_n)\) through train of \( \hat{D} \hat{D} \hat{D} \) using the identities \( \mathbb{D} \) & \( \mathbb{D} \) :

\( (L_0 - 1 - \hat{1}_n)_{\text{tree}} = \frac{1}{L_0 + n - 1} \hat{D} \frac{1}{L_0 + n - 1} \hat{D} \cdots (L_0 - 1 - \hat{1}_n) |0; \rho_1 \rangle = 0 \)

vanishes by on-shell condition

But these vanish by the on-shell condition.

\[ \Rightarrow \quad \hat{1}_n |\text{tree}\rangle = 0, \quad \text{for } n > 0 \]

These are the subsidiary on-shell "gauge" conditions one gets when setting \( \alpha_0 = 1 \).

So, in summary, for \( \alpha_0 = 1 \) we have for physical states:

\( (L_0 - 1)_{\text{tree}} = 0 \quad \text{"physical" on-shell condition} \)

\( \hat{L}_n |\text{tree}\rangle = 0 \quad \text{on-shell "gauge" condition} \)