Internal symmetry - Chan-Paton factors

Up to this point, we have studied scattering amplitudes involving neutral mesons - no quantum numbers have been introduced.

However we know:
- Mesonic states approximately fall into isospin multiplets (degenerate trajectories)
- Isospin symmetry also leads to relationships among scattering amplitudes involving different members of the same multiplet.

If we are to incorporate isospin symmetry into the dual model scattering amplitudes, we must do so while maintaining:

1. Cyclic symmetry, if external particles are identical
2. Factorization, and bootstrap consistency
   (external particles lie on Regge trajectory)

Furthermore, we require (from experiment)
3. Absence of poles with isospin \(\geq 1\) (no extra mesons)

Solution is surprisingly simple! — Chan-Paton

Consider pion scattering: \(\pi^a \pi^b \rightarrow \pi^c \pi^d\)

previously, we wrote: \(A_y(s,t,u) = g \left[ B_y(s,t) + B_y(s,u) + B_y(t,u) \right] \)

Chan-Paton: Multiply each term by multiplicative factor \(\text{Tr} \left[ T^a T^b T^c T^d \right] \) with ordering of \(T^\alpha\) appropriate to each term. 

\(T^0 = \frac{1}{2} \mathbb{1}, \quad T^3 = \frac{1}{2} \gamma^3 \)

It will be clear later that bootstrap consistency precludes external particles from being pions — all that is important is that these particles carry isospin.
\[ B_q(s, t) \rightarrow B_q^{abcd}(s, t) = \text{Tr} \left[ T^d T^c T^b T^a \right] B_q(s, t) \]

\[ B_q(s, u) \rightarrow B_q^{abcd}(s, u) = \text{Tr} \left[ T^c T^d T^b T^a \right] B_q(s, u) \]

\[ B_q(t, u) \rightarrow B_q^{abcd}(t, u) = \text{Tr} \left[ T^d T^a T^c T^b \right] B_q(t, u) \]

So that our new amplitude is

\[ A_q^{abcd}(s, t, u) = g \left[ B_q^{abcd}(s, t) + B_q^{abcd}(s, u) + B_q^{abcd}(t, u) \right] \]

More generally,

\[ A_N^{abc...}(s, ...) = g \left[ \sum_{\text{perm}} \frac{1}{(N-1)!} \right] \text{Tr} \left[ T^{a_1} ... T^{a_N} \right] B_N(s, ...) \]

Sum over all inequivalent planar orderings.

This very simple prescription satisfies all three requirements:

1. Cyclic symmetry follows immediately from cyclic property of trace:

\[ \text{Tr} \left[ T^{a_1} ... T^{a_N} \right] = \text{Tr} \left[ T^{a_2} T^{a_1} ... \right] \]

2. Factorization:

Recall the factorized form of \( N \)-particle amplitude (operator formalism)

\[ B_N = \sum_{\{r\}} \sum_{r_0} \frac{1}{r_0^2 - \alpha \tau_0} \langle 0 | \hat{V}(r_{N-2}) ... \hat{V}(r_3) \rangle \frac{1}{-\alpha(s_{12}) + \tau + r_0} \langle \{r_3\} ... \hat{V}(r_2) | 0 \rangle \]

\[ \equiv \sum_{\{r, r_0\}} \frac{1}{-\alpha(s_{1k}) + \tau + r_0} B_{\{r, r_0\} \rightarrow r_{N-k}} B_{k \rightarrow \{r_0, r\}} \text{ product of scattering amplit.} \]
Can demonstrate factorization of $B_{N}^{a_{1}...a_{m}}$ in the following way:

Consider the 4-point function $B_{N}^{abcd}(s,t) = \text{Tr}[T^{d}T^{c}T^{b}T^{a}] B_{N}(s,t)$.

Write $\text{Tr}\left[T^{d}T^{c}T^{b}T^{a}\right] = \text{Tr}[T^{d}T^{c}11T^{b}T^{a}11]$.

Putting in the indices,

\[ (T^{d}T^{c})_{ij}^{\phantom{ij}} \frac{1}{\lambda_{kk}} \frac{1}{\lambda_{mm}} \frac{1}{\lambda_{ii}} \]

Regroup as shown

\[ (T^{d}T^{c})_{i j}^{\phantom{ij}} \frac{1}{\lambda_{kk}} \frac{1}{\lambda_{mm}} \frac{1}{\lambda_{ii}} = \frac{1}{2} \sum_{e=0}^{2} T_{jk}^{e} T_{kn}^{e} \]

$U(2)$ group identity:

\[ U(2) \text{ group identity:} \]

Reorganize:

\[ = \frac{n}{4} \sum_{e=0}^{2} \sum_{f=0}^{2} (T^{d}T^{c})_{j i}^{\phantom{ij}} T_{e j i}^{\phantom{ij}} \frac{1}{\lambda_{kk}} \frac{1}{\lambda_{mm}} \frac{1}{\lambda_{ii}} \]

Sum over $f$: \[ = 2 \sum_{e=0}^{2} (T^{d}T^{c})_{i j}^{\phantom{ij}} T_{e j i}^{\phantom{ij}} \frac{1}{\lambda_{kk}} \frac{1}{\lambda_{mm}} \frac{1}{\lambda_{ii}} \]

\[ \therefore \text{Tr}\left[T^{d}T^{c}T^{b}T^{a}\right] = 2 \sum_{e=0}^{2} \text{Tr}\left[T^{d}T^{c}T^{e}\right] \text{Tr}\left[T^{e}T^{b}T^{a}\right] \]

Factor of 2 due to (silly)

Normalization of $T_{e}$

More generally,

\[ \text{Tr}\left[T^{a_{1}}...T^{a_{k+1}}...T^{a_{m}}\right] = 2 \sum_{b=0}^{2} \text{Tr}\left[T^{a_{1}}...T^{a_{k+1}}...T^{b}\right] \text{Tr}\left[T^{b}T^{a_{k+1}}...T^{a_{m}}\right] \]

Therefore, the factorized amplitude is:

\[ B_{N}^{a_{1}...a_{m}} = \sum_{b=0}^{2} \sum_{(r_{1},r_{3})} -\alpha(s_{bb}) + r_{+}r_{0} \left( \text{Tr}\left[T^{a_{1}}...T^{a_{k+1}}...T^{b}\right] B_{(r_{1},r_{3}) \rightarrow m-k} B_{k \rightarrow \{r_{1},r_{3}\}} \right) \]

\[ = 2 \sum_{b=0}^{2} \sum_{(r_{1},r_{3})} -\alpha(s_{bb}) + r_{+}r_{0} B_{(r_{1},r_{3}) \rightarrow m-k} B_{k \rightarrow \{r_{1},r_{3}\}} B_{a_{1}...a_{b}}^{b} \]

Note
Notice, in addition to the \( \xi, \xi_0 \) quantum numbers, poles carry isospin quantum numbers, \( T^6 \). \( \Rightarrow \) spectrum degeneracy increased 3-fold.

Finally,

\( \xi \) As can see from factorization, poles with isospin other than 0 or 1 will not appear since the \( T^a \)'s are 2×2 matrices. - representation is fixed by that of the \( T^a \)'s.
How can I see that isospin-singlet has opposite signature as isospin-triplet

\[ A_q = \sum_{n=0}^{\infty} \frac{\text{Tr}[T^d T_c T^e] R_n(-\alpha(t)) + \text{Tr}[T^c T^d T^e] R_n(-\alpha(u))}{\eta - \alpha(s)} \]

Factorize isospin structure:

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\eta=0}^{\infty} \frac{\text{Tr}[T^d T_c T^e] R_n(-\alpha(t)) + \text{Tr}[T^c T^d T^e] R_n(-\alpha(u))}{\eta - \alpha(s)} \]

\[ = \frac{1}{2} \sum_{\eta=0}^{\infty} \text{split sum into singlet (0,0) and triplet (c=1, 2, 3)} \]

\[ \text{singlet (0,0)} = \text{isospin singlet} \]

\[ \text{triplet (c=1, 2, 3)} \]

Now use \( T^c, T^d T^e = \frac{1}{2} S_{cd} + (d = 0 \text{ terms}) \). Then \( \text{Tr}[T^c T^e] = 0 \).

\[ = \frac{1}{2} \sum_{\eta=0}^{\infty} \frac{\text{Tr}[T^d T_c T^e] R_n(-\alpha(t)) + \text{Tr}[T^c T^d T^e] R_n(-\alpha(u))}{\eta - \alpha(s)} \]

Bootstrap condition:

\[ \text{External particles are } f_0 \text{ isosbroms.} \]

(Non-existent in nature \( \rightarrow \) pole is a nonsense point.)