Gauge Dependence of $T_c$ and $\phi_{\text{crit}}$

Denote finite temperature effective potential (free energy) by $V_{\text{eff}}(\phi_i, T; \xi)$

Critical temperature defined by: (not but not sufficient condition - need $\frac{\partial^2 V}{\partial \phi_i^2} > 0$)

1. $V_{\text{eff}}(\phi_i^{(1)}, T_c; \xi) - V_{\text{eff}}(\phi_i^{(2)}, T_c; \xi) = 0$

2. $\frac{\partial V_{\text{eff}}}{\partial \phi_i}(\phi_i^{(1)}, T_c; \xi) = \frac{\partial V_{\text{eff}}}{\partial \phi_i}(\phi_i^{(2)}, T_c; \xi) = 0$

$\phi_i^{(1)}, \phi_i^{(2)}$ and $T_c$ are found by simultaneously solving/inverting the above eqns.

$\Rightarrow$ obtain $\phi_i(\xi, ...)$ and $\phi_2(\xi, ...)$ and $T_c(\xi, ...)$

parameters of theory

Differentiate 1 with respect to $\xi$:

$0 = \left( \frac{\partial V_{\text{eff}}}{\partial \phi_i} \phi_i^{(1)} + \frac{\partial V_{\text{eff}}}{\partial T} \frac{\partial T_c}{\partial \xi} + \frac{\partial V_{\text{eff}}}{\partial \xi} \right) - \left( \frac{\partial V_{\text{eff}}}{\partial \phi_i} \phi_i^{(2)} + \frac{\partial V_{\text{eff}}}{\partial T} \frac{\partial T_c}{\partial \xi} + \frac{\partial V_{\text{eff}}}{\partial \xi} \right) = 0$

At $\phi = \phi_i^{(1)}$ and $\phi = \phi_i^{(2)}$,

we have $\frac{\partial V_{\text{eff}}}{\partial \phi_i} = 0$. Then by N. ident: $\frac{\partial V_{\text{eff}}}{\partial \xi} = 0$

$0 = \left( \left. \frac{\partial V_{\text{eff}}}{\partial T} \right|_{\phi_i^{(1)}} - \left. \frac{\partial V_{\text{eff}}}{\partial T} \right|_{\phi_i^{(2)}} \right) \frac{\partial T_c}{\partial \xi}$

In general doesn’t vanish

$\Rightarrow \frac{\partial T_c}{\partial \xi} = 0$

Hence, critical temperatures are indep. of gauge parameter $\xi$. 
Gauge dep. of order parameter, $\phi_i$:

Differentiate $2$ w.r.t $\xi$:

\[
\frac{\partial^2 V_{\text{eff}}}{\partial \phi_i \partial \phi_j} \frac{\partial \phi_i^{(1,2)}}{\partial \xi} + \frac{\partial^2 V_{\text{eff}}}{\partial T \partial \phi_i} \frac{\partial T_{\xi}}{\partial \xi} + \frac{\partial^2 V_{\text{eff}}}{\partial \xi \partial \phi_i} \frac{\partial \phi_i^{(1,2)}}{\partial \phi_j} = 0
\]

(established)

Use Nielsen identity:

\[
\frac{\partial V_{\text{eff}}}{\partial \xi} = -C_j(\phi_i, \xi) \frac{\partial V_{\text{eff}}}{\partial \phi_j}
\]

Differentiate:

\[
\frac{\partial^2 V_{\text{eff}}}{\partial \xi \partial \phi_i} = -\frac{\partial C_j}{\partial \phi_i} \frac{\partial V_{\text{eff}}}{\partial \phi_j} - C_j(\phi_i, \xi) \frac{\partial^2 V_{\text{eff}}}{\partial \phi_i \partial \phi_j}
\]

When evaluated at minimum, this vanishes.

So,

\[
\left( \frac{\partial \phi_i^{(1,2)}}{\partial \xi} - C_j(\phi_i, \xi) \right) \frac{\partial^2 V_{\text{eff}}}{\partial \phi_i \partial \phi_j} \bigg|_{\phi_i^{(1,2)}} = 0
\]

This is the mass-matrix (does not vanish in general)

\[
\Rightarrow \frac{\partial \phi_j^{(1,2)}}{\partial \xi} = C_j(\phi_i, \xi)
\]

as expected, since they follow characteristic curves.