Finite Temperature (Quantum) Field Theory

Preliminaries:
Working with relativistic field theory: the Minkowski metric
is \( g_{\mu \nu} = \text{diag}(-1, -1, -1, -1) \). And of course, \( c = \hbar = 1 \).

Statistical Mechanics:
The microscopic description of thermodynamics is handled by Statistical Mechanics. Here, we will set the Boltzmann factor \( k_B = 1 \), so that \( \beta = 1/k_B T = 1/T \), and \( T \) is quoted in [energy].

\[
 k_B = 8.6 \times 10^{-5} \, \text{eV} \, \text{K}^{-1}
\]

One can start from any one of the various ensembles:
- Microcanonical \( E, V, N \) fixed
- Canonical \( T, V, N \) fixed
- Grand canonical \( T, V, \mu \) fixed

All equivalent in the thermodynamic limit.

Brief review: (free field theory)

\[
 L = \frac{1}{2} \partial \mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \]

\[
 \hat{L} = \int d^3 x \left[ \frac{1}{2} \hat{\pi}^2 + \frac{1}{2} (\nabla \hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi}^2 \right]
\]

Impose CCRs,
\[
 \hat{\phi}(\mathbf{p}) = \frac{1}{\sqrt{2\omega_p}} \left( \hat{a}_\mathbf{p} + \hat{a}_\mathbf{p}^\dagger \right)
\]

\[
 \hat{\pi}(\mathbf{p}) = -i \sqrt{\frac{\omega_p}{2}} \left( \hat{a}_\mathbf{p} - \hat{a}_\mathbf{p}^\dagger \right)
\]

\[
 \Rightarrow \hat{\Pi} = \int \frac{d^3 p}{(2\pi)^3} \omega_p \left( \hat{a}_\mathbf{p}^\dagger \hat{a}_\mathbf{p} + \frac{1}{2} \left( \frac{2\pi)^3 S(3)(0)}{V} \right) \right)
\]

Volume factor \( V \), taken to be infinite.

\[
 = \frac{1}{2} V \int \frac{d^3 p}{(2\pi)^3} \omega_p + \int \frac{d^3 p}{(2\pi)^3} \omega_p \left( \hat{a}_\mathbf{p}^\dagger \hat{a}_\mathbf{p} \right) = V\varepsilon_{\text{vac}} + \int \frac{d^3 p}{(2\pi)^3} \omega_p \left( \hat{a}_\mathbf{p}^\dagger \hat{a}_\mathbf{p} \right)
\]

Vacuum energy

\[
 \frac{\hat{\Pi}}{V} = \hat{\Pi} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_p + \frac{1}{V} \int \frac{d^3 p}{(2\pi)^3} \omega_p \left( \hat{a}_\mathbf{p}^\dagger \hat{a}_\mathbf{p} \right)
\]

"Hamiltonian Density"
Normalization of States in Thermal Field Theory

According to P&S convention (the one I'm using), the plane wave expansion for the scalar field is:

\[
\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left( \hat{a}_p e^{-ip \cdot x} + \hat{a}^+_p e^{ip \cdot x} \right)
\]

By dimensional analysis:

\[ [E]^1 = [E]^3 \times [E]^{-1/2} \]

The ladder operators, \( \hat{a}_p \) & \( \hat{a}^+_p \), have dimensions \([E]^{-3/2}\) (same units as \( \sqrt{E} \)).

Normalization of multiparticle states:

Vacuum State: \( |0\rangle \) \[ \langle 0|0\rangle = 1 = [E]^0 \text{ units} \]

One-particle State: \( |\vec{P}\rangle = \sqrt{2\omega_p} \hat{a}^+_p |0\rangle \) \[ \langle \vec{P}|\vec{P}\rangle = 2\omega_p (2\pi)^3 \delta^{(3)}(\vec{P} - \vec{P}) = 2\omega_p \sqrt{E} = [E]^{-2} \]

Two-particle State: \( |\vec{P}_1, \vec{P}_2\rangle = \sqrt{2\omega_{p_1}} \sqrt{2\omega_{p_2}} \hat{a}^+_p \hat{a}^+_q |0\rangle \) \[ \langle \vec{P}_1, \vec{P}_2|\vec{P}_1, \vec{P}_2\rangle = [E]^{-4} \]

... that the normalization of multiparticle (Fock) states have differing dimensions causes problems. Consider the calculation of the partition function:

\[
Z(\beta) = \text{Tr} \left[ e^{-\beta \hat{H}} \right] = \sum_{\text{states, } \phi} \langle \phi | \phi \rangle e^{-\beta E_{\phi}}
\]

\[ = \langle 0|0\rangle e^{-\beta E_{00}} + \langle \#|\vec{P}\rangle e^{-\beta E_{\vec{P}}} + \langle \#|\vec{P}_1, \vec{P}_2|\vec{P}_1, \vec{P}_2\rangle e^{-\beta E_{\vec{P}_1 \vec{P}_2}} + \ldots
\]

\[ = e^{-\beta E_{00}} + \langle \# \rangle (2\omega_p \sqrt{E} e^{-\beta E_{00}}) + \langle \# \rangle (2\omega_{p_1} \omega_{p_2}) (2\omega_p) \frac{\sqrt{E}}{[E]^{-2}} e^{-\beta E_{\vec{P}_1 \vec{P}_2}} + \ldots
\]

\[ \times \text{Dimensions!}
\]

The problem (obviously) lies in how the Fock states were normalized — chosen to be Lorentz invariant. \( 2\omega_p \sqrt{E} \) is a L. inv. combination.

\[ \text{Bond} \quad 2\omega_p \sqrt{E} \langle \frac{\sqrt{E}}{[E]^{-2}} \rangle = 2\omega_p \sqrt{E} \]

The end expression
Resolution: Introduce a new set of multiparticle states with a different choice of normalization. FIRST, move to box-normalization (countable modes).

Vacuum State: \( |0\rangle \rightarrow \langle 0 | 0 \rangle = 1 \) (unitless).

One-particle State: \( |1_p\rangle = \frac{1}{\sqrt{V}} \hat{a}^\dagger_\mu |0\rangle \rightarrow \langle 1_p | 1_p \rangle = \left( \frac{1}{\sqrt{V}} \right)^2 V \delta_{p_p^\prime} \langle 0 | 0 \rangle \)

Two-particle State: \( |2_{p_p^\prime}\rangle = \frac{1}{\sqrt{V}} \hat{a}^\dagger_\mu \hat{a}^\dagger_\nu |0\rangle \rightarrow \langle 2_{p_p^\prime} | 2_{p_p^\prime} \rangle = 1 \) (unitless).

This way, all multiparticle states have unit normalization (and dimensionless), making calculations more respectable.

The complication now arises when moving back to continuum normalization:

\[
\langle 1_p | 1_p \rangle = \langle 0 | \frac{1}{\sqrt{V}} \delta_\mu \frac{1}{\sqrt{V}} \hat{a}^\dagger_\nu |0\rangle = \left( \frac{1}{\sqrt{V}} \right)^2 V \delta_{p_p^\prime}
\]

\[
= \delta_{p_p^\prime} \quad \text{continuum} \rightarrow \frac{1}{V} \frac{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}^\prime)}{i}.
\]

This is usually not a problem since, in most calculations, there will be a summation over modes, \( \sum_p \), that will kill the Kronecker Delta \( \delta_{p_p^\prime} \).

Neither the P+T normalization nor the "dimensionless" normalization is a more physical normalization than the other; remember, the Hilbert space spanned by the states are rays – not vectors. The choice of normalization is purely for convenience.

\[\text{N.B. } \langle \hat{O} \rangle = \frac{\langle \psi | \hat{O} | \psi \rangle}{\langle \psi | \psi \rangle} \quad \text{Normalization convention cancels}.
\]

\[\text{expectation value of observable } \hat{O}.\]
Multiparticle excitations with a finite number of particles have an energy density that is infinitesimally larger than the vacuum energy density because of the $1/v$ factor. To appreciably change the energy density of the system, we need an "infinite" number of particles, so that there is a finite density of particles. — Can get this by having finite temperature.

If we have a box of volume $V$ inside a heat bath, the box will equilibrate to a thermal state — not the vacuum.

**Canonical Ensemble:**

Usually convenient to work in the canonical ensemble for QFT.

Thermodynamics follow from the partition function $Z(\beta) = \sum_{\text{states}, n} e^{-\beta E_n} = \sum_n \langle n | e^{-\beta \hat{H}} | n \rangle = \text{Tr}[e^{-\beta \hat{H}}]$, where $\beta(\text{equiv.}) \propto \text{"thermal density operator"}$

Easier to calculate for scalar free field theory in a finite box $\Rightarrow$ discrete modes

$$\int \frac{d^3k}{(2\pi)^3} \leftrightarrow \frac{1}{V} \sum_{\text{modes}, i}$$

$$\hat{H} = V E_{\text{vac}} + \frac{1}{V} \sum_{\text{modes}, i} \omega_{p_i} \left( \hat{a}_{p_i} \hat{a}_{p_i} + \hat{a}_{p_i}^\dagger \hat{a}_{p_i}^\dagger \right)$$

Note: $[\hat{a}_{p_i}, \hat{a}_{p_i}^\dagger] = \mathcal{O}(\delta(p_i - p_j))$

Easiest to calculate $\Rightarrow$ discrete modes

$$\hat{H} \begin{pmatrix} n_1, n_2, \ldots \end{pmatrix} = \left( V E_{\text{vac}} + \sum_i \omega_{p_i} n_i + \omega_{p_2} n_2 + \ldots \right) \begin{pmatrix} n_1, n_2, \ldots \end{pmatrix}$$

**Organization of states:**

Since the modes are countable, put all of them in a list $\{p_i\}$, $i = 1, 2, 3, \ldots$ & identify each state by occupation numbers of each mode: $|n_1, n_2, n_3, \ldots \rangle$

Then $\hat{H} \begin{pmatrix} n_1, n_2, \ldots \end{pmatrix} = \left( V E_{\text{vac}} + \sum_i \omega_{p_i} n_i + \omega_{p_2} n_2 + \ldots \right) \begin{pmatrix} n_1, n_2, \ldots \end{pmatrix}$
So,

\[ Z(\beta) = \sum_{n_1, n_2, \ldots} \langle n_1, n_2, \ldots \mid e^{-\beta \hat{H}} \mid n_2, n_2, \ldots \rangle \]

\[ = \sum_{n_1, n_2, \ldots} e^{-\beta (V E_{\text{vac}} + \Omega_{\phi_1} n_1 + \Omega_{\phi_2} n_2 + \ldots)} \]

\[ = e^{-\beta V E_{\text{vac}}} \left( \sum_{n_1=0}^{\infty} e^{-\beta \Omega_{\phi_1} n_1} \right) \left( \sum_{n_2=0}^{\infty} e^{-\beta \Omega_{\phi_2} n_2} \right) \ldots \]

geometric series

\[ = \frac{1}{1 - e^{-\beta \Omega_{\phi_1}}} \]

\[ = e^{-\beta V E_{\text{vac}}} \prod_{\text{modes}, i} \left( \frac{1}{1 - e^{-\beta \Omega_{\phi_i}}} \right) \]

Partition function for free scalar field theory.

The associated thermodynamic potential is the free energy, \( F(T, V) \), defined by:

\[ F(T, V) = -\frac{1}{\beta} \ln Z(\beta, V) \]

Then,

\[ F(T, V) = -\frac{1}{\beta} \ln \left( e^{-\beta V E_{\text{vac}}} \prod_{\text{modes}, i} \left( \frac{1}{1 - e^{-\beta \Omega_{\phi_i}}} \right) \right) \]

\[ = -\frac{1}{\beta} \left( -\beta V E_{\text{vac}} + \sum_{\text{modes}, i} \ln \left( \frac{1}{1 - e^{-\beta \Omega_{\phi_i}}} \right) \right) \]

\[ = V E_{\text{vac}} + \frac{1}{\beta} \sum_{\text{modes}, i} \ln \left( 1 - e^{-\beta \Omega_{\phi_i}} \right) \]

As \( V \to \infty \), we have \( \frac{1}{V} \sum \to \int \frac{d^3p}{(2\pi)^3} \)

\[ F(T, V) = V E_{\text{vac}} + \frac{V}{\beta} \int \frac{d^3p}{(2\pi)^3} \ln \left( 1 - e^{-\beta \sqrt{p^2 + m^2}} \right) \]

\[ \frac{F(T, V)}{V} \equiv \mathcal{F}(T) = E_{\text{vac}} + \frac{1}{\beta} \int \frac{d^3p}{(2\pi)^3} \ln \left( 1 - e^{-\beta \sqrt{p^2 + m^2}} \right) \]

Free energy density for free scalar field theory.
The integral cannot be done in closed form – make a high temperature (low \( \beta \)) expansion:

\[
\frac{F(T, V)}{V} = F(T) = E_{\text{vac}} + \frac{1}{\beta} \left[ \sum_{n=1}^{\infty} \frac{m^{2}}{2n^{2}} \ln \left( 1 - e^{-\beta \sqrt{n^{2}m^{2} + 1}} \right) \right]
\]

\( \beta m \ll 1 \): (series in \( m \))

\[
= E_{\text{vac}} + \frac{1}{\beta} \int \frac{d^{3}p}{(2\pi)^{3}} \ln \left( 1 - e^{-\beta \sqrt{p^{2} + m^{2}}} \right) + \frac{m^{2}}{2|p|} \frac{e^{-\beta |p|}}{1 - e^{-\beta |p|}} + \ldots
\]

\[
= E_{\text{vac}} + \frac{1}{\beta} \int d\Omega \int d|p| \sqrt{|p|^{2}} \ln \left( 1 - e^{-\beta |p|} \right) + \frac{m^{2}}{2|p|} \frac{e^{-\beta |p|}}{1 - e^{-\beta |p|}} + \ldots
\]

\[
= E_{\text{vac}} + \frac{1}{\beta} \frac{4\pi}{(2\pi)^{3}} \left[ -\frac{\pi^{2}}{45} + \frac{m^{2}a^{2}}{12} + \ldots \right]
\]

\( F(T) = E_{\text{vac}} + \frac{1}{\beta^{4}} \left( -\frac{\pi^{2}}{90} + \frac{(\beta m)^{2}}{24} + \ldots \right) \)

Leading term \( \frac{-\pi^{2}}{90} \frac{1}{\beta^{4}} = -\frac{\pi^{2}T^{4}}{90} \) is relevant in massless case.

- Black body radiation, Stefan's Law.

For Fermions, the sum \( \sum_{n=0}^{\infty} e^{-\beta \sqrt{n^{2}m^{2}} \rho_{f}} \) stops at \( n = 1 \):

\( 1 + e^{-\beta \sqrt{m^{2}} \rho_{f}} \)

(and also polarizations must be accounted for).

With the partition function, and hence, the free energy, we are ready to calculate thermodynamic quantities, such as internal energy \( E \), pressure \( P \),...

\[
\langle E \rangle = \frac{2}{\beta} \beta F(\beta) = \frac{\pi^{2}}{30\beta^{4}} - \frac{m^{2}}{24\beta^{2}} + \ldots
\]

Thermal potential density via Legendre transformation.

(Helmholtz energy)

\[
A(\phi, T) = F(j, T) + j \phi (d^{3}x)
\]