Last time: We set out to perturbatively calculate the n-point thermal correlation function in an interacting theory,

\[ G_{\beta}^{[n]}(\tau_1, \ldots, \tau_n) = \left\langle T(\phi_1(\tau_1), \ldots, \phi_n(\tau_n)) \frac{\tilde{S}(\beta)}{\epsilon} \right\rangle_{\beta}^{\text{Non-int}} \]

by expanding the exponential in \( \tilde{S}(\beta) \):

\[ \tilde{S}(\beta) = \mathcal{T} \left[ 1 - \int_0^\beta dx \tilde{H}_1'(v) + \frac{1}{2} \int_0^\beta dx \int_0^\beta dx' \tilde{H}_1'(v) \tilde{H}_1'(v') + \ldots \right] \]

Then \( G_{\beta}^{[n]} \) has the form:

\[ G_{\beta}^{[n]}(\tau_1, \ldots, \tau_n) = \frac{1}{\tilde{S}(\beta)_{\beta}^{\text{Non-int}}} \left\langle T(\ldots\ldots) \right\rangle_{\beta}^{\text{Non-int}} \]

This involves a product of fields, and each one contains \( \hat{\phi} \) or \( \hat{\phi}^\dagger \).

In zero-temperature field theory, \( \hat{\phi} \)'s were pushed to the left, and \( \hat{\phi}^\dagger \)'s to the right using commutation relations — automated by Wick's theorem:

\[ \hat{\phi}(x) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{\sqrt{2wp}} \left( \hat{\phi}^\dagger p e^{-ipx} + \hat{\phi}^\dagger p e^{ipx} \right) \]

Define:

\[ \hat{\phi}^+(x) + \hat{\phi}^-(x) = \hat{\phi}(x) \]

Normal-ordered product of operators \( \hat{\phi}^1 \hat{\phi}^2 \ldots \hat{\phi}_N \), written \( N(\hat{\phi}^1 \hat{\phi}^2 \ldots \hat{\phi}_N) \), means all \( \hat{\phi}^+ \)'s to the right & all \( \hat{\phi}^- \)'s to the left.

Then, \( T(\hat{\phi}(x_1) \hat{\phi}(x_2)) = N(\hat{\phi}(x_1) \hat{\phi}(x_2) + \hat{\phi}^+(x_2) \hat{\phi}^-(x_1)) \),

where \( \hat{\phi}(x_1) \hat{\phi}(x_2) = \theta(t_1 - t_2) \left[ \hat{\phi}^+(x_1), \hat{\phi}^-(x_2) \right] + \theta(t_2 - t_1) \left[ \hat{\phi}^+(x_2), \hat{\phi}^-(x_1) \right] \)

"contraction"

\[ G_{\beta}^{[2]}(x_1, x_2) = \begin{cases} G_{\beta}^{[2]}(x_1, x_2) & \text{if } t_1 < t_2 \leq \beta \\ G_{\beta}^{[2]}(x_2, x_1) & \text{if } t_2 < t_1 \leq \beta \\ G_{\beta}^{[2]}(x_1, x_2) & \text{if } t_1 < t_1 \leq \beta \end{cases} \]

\[ = G_F(x_1 - x_2) \quad (\text{Feynman Green's function}) \]

- zero temperature propagator.
(From now, \( \hat{\phi}_2 \) is short for \( \hat{\phi}(x_2) \). [In thermal field theory, \( \hat{\phi}_1 = \hat{\phi}(x_1, T_{1}) \)]

Then, for an arbitrary number of fields, Wick’s theorem states (for bosonic fields)

\[
T(\phi_1 \phi_2 \ldots \phi_N) = N(\phi_1 \phi_2 \ldots \phi_N) + \sum_{\text{single contractions}} \phi_1 \phi_2 N(\phi_3 \phi_4 \ldots \phi_N) + \sum_{\text{double contractions}} \phi_1 \phi_2 \phi_3 \phi_4 N(\ldots \phi_N) + \text{etc.}
\]

The advantage of this is that vacuum expectation values of normal ordered products vanish, \( \langle 0| N(\phi_1 \phi_2 \ldots \phi_N)|0\rangle = 0 \). So zero temperature \( n \)-point Green’s functions in a non-interacting theory decomposes into a sum of the products of 2-point Green’s functions.

However, thermal ensemble averages of normal-ordered product is in general non-vanishing

\[
\langle N(\phi_1 \phi_2 \ldots \phi_N) \rangle_{\beta} = \frac{1}{Z(\beta)} \sum_{n} e^{-\beta E_n} \langle n| \phi^{-} \ldots \phi^{+} \ldots |n\rangle \neq 0.
\]

This can render perturbative calculations of finite temperature intractable.

\rightarrow Fix this problem following the outline:

1. Define a new split \( \phi = \phi^{+} + \phi^{-} \)
   - Define a new “normal ordering”, and new contraction.
2. Modify Wick’s theorem (Generating Functional)
3. Demand \( \langle N_{\text{new}}(\phi_1 \phi_2) \rangle_{\beta} = 0 \) \( \Rightarrow \) Fixes split in Step 1.
4. Hope that \( \langle N_{\text{new}}(\phi_1 \phi_2 \ldots \phi_N) \rangle_{\beta} = 0 \) for arbitrary number of fields.
1. Instead of splitting the fields into positive and negative frequency parts, split the field operator arbitrarily. Restrict to splits linear in $\hat{\phi}$ & $\hat{\phi}^\dagger$.

Most general split:

$$\hat{\phi}^+(x, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[ (1-f_p) \hat{\phi}_p e^{-\omega_p t + i\mathbf{p} \cdot \mathbf{x}} + g_p \hat{\phi}^\dagger_p e^{\omega_p t - i\mathbf{p} \cdot \mathbf{x}} \right]$$

$$\hat{\phi}^-(x, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[ f_p \hat{\phi}_p e^{-\omega_p t + i\mathbf{p} \cdot \mathbf{x}} + (1-g_p) \hat{\phi}^\dagger_p e^{\omega_p t - i\mathbf{p} \cdot \mathbf{x}} \right]$$

$f_p$ and $g_p$ are c-numbers (generally different for each mode) parametrizing the split. Straightforward to show $\hat{\phi} = \hat{\phi}^+ + \hat{\phi}^-$.

Define the new normal ordering such that all $\hat{\phi}^+$'s are still to the right, all $\hat{\phi}^-$'s still to the left. For two fields,

$$N(\hat{\phi}_1, \hat{\phi}_2) = N((\hat{\phi}_1^+ + \hat{\phi}_1^-)(\hat{\phi}_2^+ + \hat{\phi}_2^-))$$

$$= \hat{\phi}_1^+ \hat{\phi}_2^+ + \hat{\phi}_2^- \hat{\phi}_1^+ + \hat{\phi}_1^- \hat{\phi}_2^+ + \hat{\phi}_1^- \hat{\phi}_2^-$$

Note: With the above arbitrary split, new normal ordering is not symmetric, $N(\hat{\phi}_1, \hat{\phi}_2) \neq N(\hat{\phi}_2, \hat{\phi}_1)$.

Then define the new contraction

$$\overline{\phi}_1 \phi_2 \equiv T(\hat{\phi}_1 \hat{\phi}_2) - N(\hat{\phi}_1 \hat{\phi}_2)$$

[so that $T(\hat{\phi}_1 \hat{\phi}_2) = N(\hat{\phi}_1 \hat{\phi}_2) + \overline{\phi}_1 \phi_2$]

Of course, the new contraction will look different — can work it out using the known def of $T$-ordering, and the above def of $N$-ordering.
\[
W(h, \hat{\phi}_1, \hat{\phi}_2) = \Theta(t_1 - t_2) \hat{\phi}_1 \hat{\phi}_2 \pm \Theta(t_2 - t_1) \hat{\phi}_2 \hat{\phi}_1
\]

\[
= \Theta(t_1 - t_2) \left( \hat{\phi}_1^+ \hat{\phi}_2^+ + \hat{\phi}_1^+ \hat{\phi}_2^- + \hat{\phi}_1^- \hat{\phi}_2^+ + \hat{\phi}_1^- \hat{\phi}_2^+ \right)
\pm \Theta(t_2 - t_1) \left( \hat{\phi}_2^+ \hat{\phi}_1^+ + \hat{\phi}_2^+ \hat{\phi}_1^- + \hat{\phi}_2^- \hat{\phi}_1^+ + \hat{\phi}_2^- \hat{\phi}_1^- \right)
\]

and
\[
N(h, \hat{\phi}_1, \hat{\phi}_2) = \Theta(t_1 - t_2) \left( \hat{\phi}_1^+ \hat{\phi}_2^+ \pm \hat{\phi}_2^- \hat{\phi}_1^+ + \hat{\phi}_2^- \hat{\phi}_1^- + \hat{\phi}_2^- \hat{\phi}_2^- \right)
+ \Theta(t_2 - t_1) \left( \text{ditto} \right)
\]

The two cases collapse into one.

Subtracting the two, \( \hat{\phi}_1 \hat{\phi}_2 = W(h, \hat{\phi}_1, \hat{\phi}_2) - N(h, \hat{\phi}_2) \)

\[
\hat{\phi}_1 \hat{\phi}_2 = \Theta(t_1 - t_2) \left( \hat{\phi}_1^+ \hat{\phi}_2^+ + \hat{\phi}_1^+ \hat{\phi}_2^- + \hat{\phi}_1^- \hat{\phi}_2^+ + \hat{\phi}_1^- \hat{\phi}_2^- \right)
\pm \Theta(t_2 - t_1) \left( \hat{\phi}_2^+ \hat{\phi}_1^+ + \hat{\phi}_2^+ \hat{\phi}_1^- + \hat{\phi}_2^- \hat{\phi}_1^+ + \hat{\phi}_2^- \hat{\phi}_1^- \right)
\]

\[
- \Theta(t_1 - t_2) \left( \hat{\phi}_1^+ \hat{\phi}_2^+ + \hat{\phi}_1^+ \hat{\phi}_2^- + \hat{\phi}_1^- \hat{\phi}_2^+ + \hat{\phi}_1^- \hat{\phi}_2^- \right)
- \Theta(t_2 - t_1) \left( \hat{\phi}_2^+ \hat{\phi}_1^+ + \hat{\phi}_2^+ \hat{\phi}_1^- + \hat{\phi}_2^- \hat{\phi}_1^+ + \hat{\phi}_2^- \hat{\phi}_1^- \right)
\]

\[
\hat{\phi}_1 \hat{\phi}_2 = \Theta(t_1 - t_2) \left[ \hat{\phi}_1^+, \hat{\phi}_2^- \right]
+ \Theta(t_2 - t_1) \left[ \hat{\phi}_2^+, \hat{\phi}_1^+ \right] + \left[ \hat{\phi}_2^+, \hat{\phi}_1^- \right] + \left[ \hat{\phi}_2^-, \hat{\phi}_1^- \right])
\]

Notice a) Not symmetric \( \hat{\phi}_1 \hat{\phi}_2 \neq \hat{\phi}_2 \hat{\phi}_1 \) unless \( \left[ \hat{\phi}_2^+, \hat{\phi}_1^+ \right] + \left[ \hat{\phi}_2^-, \hat{\phi}_1^- \right] = 0 \).

b) Recall that the commutator of two operators \( [\hat{A}, \hat{B}] \) is in general another non-commuting operator \( \hat{C} \). This would imply the contraction \( \hat{\phi}_1 \hat{\phi}_2 \) is an operator, which is undesirable. But the requirement \( h \) splits linear in \( \hat{\alpha} \) and \( \hat{\alpha}^\dagger \) guarantees that \( \hat{\phi}_1 \hat{\phi}_2 \) will be a commuting number since

\[
[\hat{\phi}_1^+, \hat{\phi}_1^-] \sim [\hat{\alpha}_1, \hat{\alpha}_1^\dagger] = (2\pi)^3 \delta^3(-)
\]

(n.b. \( [\hat{\alpha}^\dagger \hat{\alpha}, \hat{\alpha}^\dagger] \neq c\text{-number} \))
2. With new splits, normal ordering and contractions, Wick's theorem which was

\[ T(\phi_1 \phi_2 \cdots \phi_N) = N(\phi_1 \phi_2 \cdots \phi_N) + \sum \frac{1}{2} \phi_1 \phi_2 N(\phi_3 \phi_4 \cdots \phi_N) \]

\[ + \sum \frac{1}{4} \phi_1 \phi_2 \phi_3 \phi_4 N(\phi_5 \phi_6 \cdots \phi_N) + \text{etc...} \]

might have to be modified. As long as contractions are c-numbers (splits linear in \( \hat{a} \& \hat{a}^\dagger \)) this is not the case.

- Could prove this using induction – textbook strategy, but calculation becomes exceedingly hairy. Instead, prove by using generating functionals:

**Identities to remember:**

1. \( e^{\alpha \hat{A}} e^{\beta \hat{B}} = e^{\alpha \hat{A} + \beta \hat{B}} e^{\frac{1}{2} \alpha \beta [\hat{A}, \hat{B}]} \) \( \forall [\hat{A}, \hat{B}] = \text{c-number} \) (Baker–Campbell–Hausdorff)

2. \( T(\hat{e}^{\hat{\theta}(\tau_1)} + \hat{e}^{\hat{\theta}(\tau_2)}) = e^{\theta(\tau_1-\tau_2)} e^{\hat{\theta}(\tau_1)} e^{\hat{\theta}(\tau_2)} + \theta(\tau_2-\tau_1) e^{\hat{\theta}(\tau_2)} e^{\hat{\theta}(\tau_1)} \)

   \( \text{means to Taylor expand exponentials first, then apply time-ordering to each term.} \)

   Similarly,

3. \( N(\hat{e}^{\hat{\phi}^+_1 + \hat{\phi}^-_2}) = N(1 + (\hat{\phi}^+_1 + \hat{\phi}^-_2) + \frac{1}{2!} (\hat{\phi}^+_1 \hat{\phi}^+_2 + \hat{\phi}^-_1 \hat{\phi}^-_2 + \hat{\phi}^-_1 \hat{\phi}^+_2 + \hat{\phi}^+_1 \hat{\phi}^-_2) + \ldots) = e^{\hat{\phi}^+_1} e^{\hat{\phi}^-_2} \)

**Proof:**

Consider \( T(e^{-i \int d\tau \int d^3x \, J(x, \tau), \hat{\theta}(x, \tau)}) \equiv T(e^{-i \int \tau_2 - \tau_1 \, dt \, \hat{\Theta}(t)}) \)

Some arbitrary c-number function that will act as a source.

Range of integration is completely irrelevant for the proof.

abbr. \( \hat{\Theta}(\tau) = \int d^3x \, J(x, \tau), \hat{\theta}(x, \tau) \)
Split up integral into $N$ equal-time parts, and write the integral as a Riemann sum.

\[
  \int_{-T_f}^{T_f} e^{-i\int_{t_i}^{T_f} dt \hat{\delta}(t)}
  = \lim_{\Delta t \to 0} \left( \lim_{N \to \infty} T \left( e^{-i \left[ \hat{\delta}(t_i) \Delta t + \hat{\delta}(t_i + \Delta t) \Delta t + \ldots + \hat{\delta}(t_i + N \Delta t) \Delta t \right]} \right) \right)
  \]

Using identity 2, split up exp, and place latest time to the left.

\[
  = \lim_{\Delta t \to 0} T \left( e^{-i \hat{\delta}(t_N) \Delta t} e^{-i \hat{\delta}(t_{N-1}) \Delta t} \ldots e^{-i \hat{\delta}(t_0) \Delta t} \right)
  \]

Combine first two exponential factors using identity 1.

\[
  = \lim_{\Delta t \to 0} \left( e^{-i \left( \hat{\delta}(t_N) + \hat{\delta}(t_{N-1}) \right) \Delta t} - \frac{1}{2} (\Delta t)^2 \left[ \hat{\delta}(t_N), \hat{\delta}(t_{N-1}) \right] \right) \ldots e^{-i \hat{\delta}(t_0) \Delta t}
  \]

Combine next two terms

\[
  = \lim_{\Delta t \to 0} \left( e^{-i \left( \hat{\delta}(t_N) + \hat{\delta}(t_{N-1}) + \hat{\delta}(t_{N-2}) \right) \Delta t} - \frac{1}{2} (\Delta t)^2 \left[ \hat{\delta}(t_N), \hat{\delta}(t_{N-1}), \hat{\delta}(t_{N-2}) \right] \right) \ldots e^{-i \hat{\delta}(t_0) \Delta t} - \frac{1}{2} (\Delta t)^2 \left[ \hat{\delta}(t_N), \hat{\delta}(t_{N-1}) \right]
  \]

But $[\hat{\delta}(t_N), \hat{\delta}(t_{N-1}), \hat{\delta}(t_{N-2})] = [\hat{\delta}(t_N), \hat{\delta}(t_{N-2})] + [\hat{\delta}(t_{N-1}), \hat{\delta}(t_{N-2})]$.

Both c-numbers, keep combining all the exponentials, and open up all the commutators that appear. In the end, we get

\[
  = \lim_{\Delta t \to 0} \left( e^{-i \left( \hat{\delta}(t_N) + \ldots + \hat{\delta}(t_0) \right) \Delta t} - \frac{1}{2} (\Delta t)^2 \sum_{j=1}^{N-1} [\hat{\delta}(t_j), \hat{\delta}(t_i)] \right)
  \]

$\Delta t \sum_i \Delta t \sum_j^{\infty}$
Take the $\Delta z \to 0$. Mind the double sum in the second exponential.

$\Delta t \sum_{i,j} \Delta z \sum_{j} \rightarrow \int d\tau_x \int d\tau_y \theta(\tau_y - \tau_x)$

$T(e^{-i\int d^4x J(x) \hat{\phi}(x)}) = e^{-i\int d^4x J(x) \hat{\phi}(x)} e^{-\frac{1}{2} \int d^4x \int d^4y \theta(\tau_y - \tau_x) [\hat{\phi}(\tau_y), \hat{\phi}(\tau_x)]}$

or

$T(e^{-i\int d^4x J(x) \hat{\phi}(x)}) = e^{-i\int d^4x J(x) \hat{\phi}(x)}$ 

$x e^{-\frac{1}{2} \int d^4x \int d^4y \theta(\tau_y - \tau_x) J(x) J(y) [\hat{\phi}(\tau_y), \hat{\phi}(\tau_x)]}$

$dt_x = d^3x \ dt_x$

Write the first factor in R.H.S as

$e^{-i\int d^4x J(x) \hat{\phi}(x)} = e^{-i\int d^4x J(x) (\hat{\phi}^{-}(x) + \hat{\phi}^{+}(x))}$

Use identity 1) (with $\hat{\phi}^{-} = \phi^{-} - \hat{\phi}^{-}$) to split up exponentials.

$= e^{-i\int d^4x J(x) \hat{\phi}^{-}(x)} - i\int d^4x J(x) \hat{\phi}^{+}(x) (\hat{\phi}^{-})^2 \frac{1}{2} \int d^4x \int d^4y J(x) J(y) [\hat{\phi}^{+}(x), \hat{\phi}^{-}(y)]$

This is normal ordered. Use iden 2)

$= N(e^{-i\int d^4x J(x) (\hat{\phi}^{-}(x) + \hat{\phi}^{+}(x))} \frac{1}{2} \int d^4x \int d^4y J(x) J(y) [\hat{\phi}^{+}(x), \hat{\phi}^{-}(y)] e^{-\frac{1}{2} \int d^4x \int d^4y J(x) J(y) [\hat{\phi}^{+}(x), \hat{\phi}^{-}(y)]}$

So, together,

$T(e^{-i\int d^4x J(x) \hat{\phi}(x)}) = N(e^{-i\int d^4x J(x) \hat{\phi}(x)})$

$x e^{-\frac{1}{2} \int d^4x \int d^4y J(x) J(y) (\theta(\tau_y - \tau_x) [\hat{\phi}(\tau_y), \hat{\phi}(\tau_x)] + [\hat{\phi}^{+}(x), \hat{\phi}^{-}(y)]}$

Simplify this.

"this" = $\theta(\tau_y - \tau_x) [\hat{\phi}(\tau_y), \hat{\phi}(\tau_x)] + [\hat{\phi}^{+}(x), \hat{\phi}^{-}(y)] + [\hat{\phi}^{-}(\tau_y), \hat{\phi}^{+}(\tau_x)] + [\hat{\phi}^{+}(\tau_y), \hat{\phi}^{+}(\tau_x)]$

+ $\theta(\tau_x - \tau_y) [\hat{\phi}^{+}(\tau_x), \hat{\phi}^{-}(\tau_y)] + \theta(\tau_y - \tau_x) [\hat{\phi}^{+}(\tau_x), \hat{\phi}^{-}(\tau_y)]$
\[ \text{this} = \Theta(t_y - t_x) \left[ \hat{\phi}^+(x), \hat{\phi}^-(y) \right] \]

\[ + \Theta(t_y - t_x) \left( \left[ \hat{\phi}^+(y), \hat{\phi}^+(x) \right] + \left[ \hat{\phi}^+(y), \hat{\phi}^-(x) \right] + \left[ \hat{\phi}^-(y), \hat{\phi}^+(x) \right] \right) \]

\[ = \phi(x) \phi(y) \]

So finally,

\[ T \left( e^{-i\int d^4x J(x) \hat{\phi}(x)} \right) = N \left( e^{-i\int d^4x J(x) \phi(x)} \right) e^{-\frac{i}{2} \int d^4x \int d^4y J(x) J(y) \phi(x) \phi(y)} \]

This is Wick's theorem in compact form. Take functional derivatives with respect to the source \( J(x, t) \) to drop desired number of \( \phi \)'s down.

Then take \( J(x, t) \to 0 \).

- True for any splitting of the field \( \hat{\phi}(x, t) \), as long as the split is linear in \( \phi \) & \( \delta \phi \), implying \( \phi \delta \phi \) is a c-number.

Derivation is independent of the thermal density operator. Hence Wick's theorem is applicable in systems that are out of thermal equilibrium, described by a different \( \hat{\rho} \).
3. Recall \( T(\hat{\phi}_1 \hat{\phi}_2) = N(\hat{\phi}_1 \hat{\phi}_2) + \phi_1 \phi_2 \). Take thermal average:

\[
\Rightarrow \quad \langle T(\hat{\phi}_1 \hat{\phi}_2) \rangle_\beta = \langle N(\hat{\phi}_1 \hat{\phi}_2) \rangle_\beta + \langle \phi_1 \phi_2 \rangle_\beta
\]

Already calculated

Require this to vanish

\( = \phi_1 \phi_2 \) (just a c-number)

Then

\[
\langle T(\hat{\phi}_1 \hat{\phi}_2) \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \frac{1}{2wp} \left[ \left( n_b(\omega_p) + 1 \right) e^{-\omega_p |\tau| + ip \cdot \vec{x}} + n_b(\omega_p) e^{\omega_p |\tau| - ip \cdot \vec{x}} \right]
\]

where \( \tau = \tau_1 - \tau_2 \), \( \vec{x} = \vec{x}_2 - \vec{x}_1 \)

One can fix \( f_p \) & \( g_p \) (hence the split) directly by calculating \( \langle N(\hat{\phi}_1 \hat{\phi}_2) \rangle_\beta \) and setting the result to zero—not necessary, but instructive.

\[
\langle N(\hat{\phi}_1 \hat{\phi}_2) \rangle_\beta = \langle \phi_1^+ \phi_2^+ \rangle_\beta + \langle \phi_2^- \phi_1^+ \rangle_\beta + \langle \phi_1^- \phi_2^+ \rangle_\beta + \langle \phi_1^- \phi_2^- \rangle_\beta = 0
\]

\[
\langle \phi_1^+ \phi_2^+ \rangle_\beta = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2wp} \left[ (1-f_p) g_p \left( n_b(\omega_p) + 1 \right) e^{-wp (\tau_1 - \tau_2) + ip \cdot (\vec{x}_1 - \vec{x}_2)} + (1-f_p) g_p n_b(\omega_p) e^{wp (\tau_1 - \tau_2) - ip \cdot (\vec{x}_1 - \vec{x}_2)} \right]
\]

\[
\langle \phi_2^- \phi_1^+ \rangle_\beta = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2wp} \left[ f_p g_p \left( n_b(\omega_p) + 1 \right) e^{-wp (\tau_1 - \tau_2) + ip \cdot (\vec{x}_1 - \vec{x}_2)} + (1-g_p) (1-f_p) \langle \phi_1^- \phi_2^+ \rangle_\beta e^{wp (\tau_1 - \tau_2) - ip \cdot (\vec{x}_1 - \vec{x}_2)} \right]
\]

By \( 2 \leftrightarrow 1 \),

\[
\langle \phi_1^- \phi_2^+ \rangle_\beta = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2wp} \left[ f_p g_p \left( n_b(\omega_p) + 1 \right) e^{-wp (\tau_1 - \tau_2) + ip \cdot (\vec{x}_1 - \vec{x}_2)} + (1-g_p) (1-f_p) n_b(\omega_p) e^{wp (\tau_1 - \tau_2) - ip \cdot (\vec{x}_1 - \vec{x}_2)} \right]
\]
\[ \langle \phi_1^- \phi_2^- \rangle_p = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\sqrt{\omega_p \omega_{p'}}} \left[ f_p(1-g_p) \langle \hat{\phi}_p^+ \hat{\phi}_p^- \rangle_p e^{-\omega_p \tau_1 + \omega_{p'} \tau_2 + ip \cdot x_2 - ip' \cdot x_2'} \\
+ f_p(1-g_p) \langle \hat{\phi}_p^+ \hat{\phi}_p^- \rangle_p e^{\omega_p \tau_1 - \omega_{p'} \tau_2 - ip \cdot x_1 + ip' \cdot x_2'} \right] \\
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ f_p(1-g_p) \left( n_b(\omega_p) + 1 \right) e^{-\omega_p \tau_1 - \omega_{p'} \tau_2 + ip \cdot (x_1 - x_2')} \right. \\
+ f_p(1-g_p) n_b(\omega_p) e^{\omega_p \tau_1 - \tau_2} - ip \cdot (x_1 - x_2') \right] = 0. \]

Then, together
\[ \langle N(\phi_1, \phi_2) \rangle_p = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ (f_p + g_p - f_p g_p + n_b(\omega_p)) e^{-\omega_p \tau_1 - \tau_2} - ip \cdot (x_1 - x_2) \right. \\
+ (f_p g_p + n_b(\omega_p)) e^{\omega_p \tau_1 - \omega_{p'} \tau_2 - ip \cdot (x_1 - x_2')} \left. \right] = 0. \]

Each Fourier component (being independent) gives a separate condition on \( f_p \) & \( g_p \):
\[
\begin{align*}
\frac{f_p}{f_p} + g_p - f_p g_p + n_b(\omega_p) &= 0 \quad &\text{Solution:} \\
\frac{g_p}{g_p} g_p + n_b(\omega_p) &= 0 \\
\Rightarrow f_p &= -n_b \pm \sqrt{n_b^2 + n_b^2}, \quad g_p = -n_b \mp \sqrt{n_b^2 + n_b^2}
\end{align*}
\]

Recall, then, the contraction:
\[
\Phi_1 \Phi_2 = \Theta(\tau_1 - \tau_2) \left[ \Phi_1^+, \Phi_2^+ \right] + \Theta(\tau_2 - \tau_1) \left[ \Phi_1^-, \Phi_2^- \right] + \left[ \Phi_1^+, \Phi_2^- \right] + \left[ \Phi_1^-, \Phi_2^+ \right]
\]

\[
\left[ \Phi_2^+, \Phi_2^+ \right] \sim \left[ \hat{\alpha}_p^+ \hat{\alpha}_p^+, \hat{\alpha}_p^- \hat{\alpha}_p^- \right] \quad \text{only cross terms non-zero commutation.}
\]
\[
\begin{align*}
\langle f_p \rangle_p &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left( 1 - f_p \right) g_p \left[ \hat{\alpha}_p^+ \hat{\alpha}_p^+ \right] e^{-\omega_p \tau_2 + \omega_{p'} \tau_1 + ip \cdot x_2 - ip' \cdot x_1} \\
+ g_p(1-f_p) \left[ \hat{\alpha}_p^+ \hat{\alpha}_p^- \right] e^{\omega_p \tau_2 - \omega_{p'} \tau_1 - ip \cdot x_2 + ip' \cdot x_1} \\
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left( 1 - f_p \right) g_p e^{\omega_p (\tau_1 - \tau_2) - ip \cdot (x_1 - x_2')} \\
- (1-f_p) g_p e^{-\omega_p (\tau_1 - \tau_2) + ip \cdot (x_1 - x_2')}
\end{align*}
\]

But \( (1-f_p) g_p = -\sqrt{n_b(1+n_b)} \), so
\[
\left[ \Phi_2^+, \Phi_1^+ \right] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ -\sqrt{n_b(1+n_b)} \sinh(\omega_p (\tau_1 - \tau_2) - ip \cdot (x_1 - x_2')) \right]
\]
Similarly,

\[ [\phi_2^+, \phi_1^-] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left( f_p(1-g_p) \left[ \hat{a}_p^\dagger, \hat{a}_p \right] e^{-\omega_p t_2 + \omega_p t_1 + i\vec{p} \cdot \vec{x}_2 - i\vec{p} \cdot \vec{x}_1} + (1-g_p) f_p \left[ \hat{a}_p^\dagger, \hat{a}_p \right] e^{\omega_p t_2 - \omega_p t_1 - i\vec{p} \cdot \vec{x}_2 + i\vec{p} \cdot \vec{x}_1} \right) \]

\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left( f_p(1-g_p) e^{\omega_p (t_1-t_2) - i\vec{p} \cdot (\vec{x}_1 - \vec{x}_2)} - f_p(1-g_p) e^{-\omega_p (t_1-t_2) + i\vec{p} \cdot (\vec{x}_1 - \vec{x}_2)} \right) \]

But \( f_p(1-g_p) = \sqrt{n_e(1+n_e)} \), so

\[ [\phi_2^-, \phi_1^-] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ \frac{1}{\sqrt{n_e(1+n_e)}} \sinh(\omega_p (t_1-t_2) - i\vec{p} \cdot (\vec{x}_1 - \vec{x}_2)) \right) \]

\[ \Rightarrow [\phi_2^+, \phi_1^-] + [\phi_2^-, \phi_1^-] = 0 \quad \text{cancel!} \]

So, even with a different split,

\[ \phi_1^+ \phi_2^- = \theta(t_1-t_2) \left[ \phi_1^+ \phi_2^- \right] + \theta(t_2-t_1) \left[ \phi_1^+ \phi_1^- \right] \]

\[ \phi_2^- \phi_1^+ = \theta(t_2-t_1) \left[ \phi_2^- \phi_1^+ \right] + \theta(t_1-t_2) \left[ \phi_2^- \phi_1^- \right] \]

\[ \text{Obviously, } \mathcal{G}_p(t_1-t_2) \quad \text{and} \quad \mathcal{G}_p(t_1-t_2) \]

\[ \text{symmetric!} \]
Strategy: To write \( N \exp(-i\mathbf{x} \cdot \mathbf{A}) = N \text{det}(-i\mathbf{x} \cdot \mathbf{J}) \) and show RHS vanishes order by order in \( \mathbf{J} \).

\[
\begin{align*}
\langle \phi | \rho | \phi \rangle & = \int dp_1 ... dp_j \sum_{p_k} \sum_{\rho} \langle \phi_{p_k} | \lambda | \phi_{p_j} \rangle \beta_i \beta_j (\text{string of Es}) \quad \text{(exponentials)}
\end{align*}
\]

\[
\begin{align*}
&= \int dp_1 dp_2 dk_1 dk_2 \langle \phi_{p_1} | \rho | \phi_{p_2} \rangle \langle \phi_{k_1} | \rho | \phi_{k_2} \rangle \left( e^{i(p_1 \cdot x_1 + i p_2 \cdot x_2 - i k_1 \cdot x_3 - i k_2 \cdot x_4) + (5 \text{ other terms})} \right)
\end{align*}
\]

Now \( \langle \phi_{p_1} | \rho | \phi_{p_2} \rangle = \text{V}^2 \left[ \frac{1}{n_{p_1} n_{p_2} n_{k_1} n_{k_2}} \left( \frac{[1]}{2} \right)^3 (p_1 \cdot x_3 + p_2 \cdot x_4) \right] \beta_i \beta_j (\text{string of Es}) \quad \text{(exponentials)}
\]

\[
\begin{align*}
&= \frac{1}{2 \pi^3} \frac{1}{2 \pi^3} \left[ \frac{1}{n_{p_1} n_{p_2} n_{k_1} n_{k_2}} \left( \frac{[1]}{2} \right)^3 (p_1 \cdot x_3 + p_2 \cdot x_4) \right] \left( 1 - \frac{1}{\sqrt{(2\pi)^3 \delta^3(p_1 \cdot p_2)}} \right)
\end{align*}
\]

Plug into \( \langle \phi | \rho | \phi \rangle \), integrate over \( k_1, k_2 \).

Integrate over \( p_2 \) in 2nd term,

\[
\begin{align*}
&= \int dp_1 dp_2 \left[ \frac{1}{2 \pi^3} \frac{1}{2 \pi^3} \left( e^{i p_1 \cdot (x_1 - x_4) + i p_2 \cdot (x_2 - x_3)} + e^{i p_1 \cdot (x_1 - x_3) + i p_2 \cdot (x_2 - x_4)} \right) \right] \left( \text{other terms} \right)
\end{align*}
\]

\[
\begin{align*}
&- \int dp_1 \frac{1}{2 \pi^3} \left[ \frac{1}{2 \pi^3} \left( e^{i p_1 \cdot (x_1 + x_2 - x_3 - x_4)} \right) \right] \cdot \cdot \cdot \quad \text{(other terms)}
\end{align*}
\]

\[
\begin{align*}
&+ \int dp_1 \frac{1}{2 \pi^3} \left[ \frac{1}{2 \pi^3} \right] \cdot \cdot \cdot \quad \text{(other terms)}
\end{align*}
\]

\[
ab + cd = (a + c)(b + d) = ab + ad + cb + cd
\]
\[
\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_B = \\
\int d^3p_1 d^3p_2 \frac{1}{2^{\omega_{p_1} \omega_{p_2}}} \eta_{p_1} \eta_{p_2} \left(e^{i \phi_1 (x_1 - x_2) + i \phi_2 (x_2 - x_3)} + e^{i \phi_1 (x_1 - x_3) + i \phi_2 (x_3 - x_4)} \right)
\]

\[
= \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2\omega_{p_1}} \eta_{p_1} \left(e^{i \phi_1 (x_1 - x_2)} + e^{-i \phi_1 (x_1 - x_2)} \right)
\times \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2\omega_{p_2}} \eta_{p_2} \left(e^{i \phi_2 (x_3 - x_4)} + e^{-i \phi_2 (x_3 - x_4)} \right)
\]

\[
+ \left[ \phi_1 (x_1 - x_2) \Rightarrow \phi_1 (x_1 - x_3) \right] + \left[ \phi_1 (x_1 - x_2) \Rightarrow \phi_1 (x_3 - x_4) \right].
\]
\[ \left\langle N \left( \sum_{n=0}^{\infty} \frac{(-i \oint J_a \phi_a)}{n!} \right)^n \right\rangle_\beta = \left\langle N \left( \sum_{n=0}^{\infty} \frac{(-\oint J_\alpha \phi_a \phi_b \phi_c \phi_d)}{(2n)!} \right)^n \right\rangle_\beta. \]

Then look at:
- Case \( n=1 \) (two fields) random: \( X_c \sim X_b, X_b \sim X_c \)
- Case \( n=2 \) (four fields)
\[ \phi = \int (dp) \sum_{\mu=1}^{2} \hat{\alpha}_{\mu}^{(\phi_1)} \hat{\alpha}_{\mu}^{(\phi_2)} e^{ip_1(x_a - x_c) + ip_2(x_b - x_d)}. \]

\[ \rho = \langle \frac{1}{(2\pi)^d} \int_a \int_b \int_c \int_d \int e^{ip_1(x_a - x_c) + ip_2(x_b - x_d)} \rangle_\beta. \]

\[ = \frac{(-1)^n}{(2n)!} \int_a \int_b \int_c \int_d \int e^{ip_1(x_a - x_c) + ip_2(x_b - x_d)} \sum_{\mu=1}^{n} \sum_{D=1}^{n} \left\langle \hat{\alpha}_{\mu_1}^{(\phi_1)} \hat{\alpha}_{\mu_2}^{(\phi_2)} \hat{\alpha}_{\mu_3}^{(\phi_3)} \hat{\alpha}_{\mu_4}^{(\phi_4)} \right\rangle_\beta \]

Need a equal number of \( \hat{\alpha}_s \) \& \( \hat{\alpha}_s \)'s. 
- Only possible structure is \( \hat{\alpha}_1^{\phi_1} \hat{\alpha}_2^{\phi_2} \hat{\alpha}_3^{\phi_3} \hat{\alpha}_4^{\phi_4} \) (n \( \hat{\alpha}_s \) \& n \( \hat{\alpha}_s \)')s

\[ \sqrt{2} \frac{1}{\bar{n}} \sum_{n_{i\nu}} \left\langle \eta_{n_1}, \ldots, n_{p_1-1}, n_{p_2-1}, n_{p_1}, \ldots, n_{p_2}, n_{p_3}, \ldots, n_{p_4} \right\rangle \]

\[ \sqrt{2} \frac{1}{\bar{n}} \sum_{n_{i\nu}} e^{-\beta \sum_{\nu \in \nu} \left( \delta_{p_{\nu}} \delta_{p_{\nu}} \right)} \left( \delta_{p_{\nu}, \delta_{p_{\nu}}} \right) \left( \delta_{\nu_{\nu}, \delta_{\nu_{\nu}}} \right) \left( \eta_{p_{\nu}} \eta_{p_{\nu}} \right) \]

\[ \rightarrow n_{\nu_1}(p_1) n_{\nu_2}(p_2) \left( \delta_{p_1, p_3} \delta_{p_2, p_4} \delta_{p_2, p_1} \delta_{p_4, p_3} \right) \]

\[ = \sum_{n_{i\nu}} \frac{(2\pi)^6}{(2\pi)^6} \left( \delta_{p_1, p_3} \delta_{p_2, p_4} \delta_{p_2, p_1} \delta_{p_4, p_3} \right) \]
\[ \times \text{(exponentials)} \]
All exponentials to read
\[ e^{ip_1(x_a-x_b) + ip_2(x_c-x_d)} \quad \text{(recall, for case } n=2) \]
for fields.

\[ \text{A In general, } 2n \text{ (field),} \]

\[ \frac{(2n)!}{h^n} \]

\[ \rightarrow n_b(p_1) n_b(p_2) e^{ip_1 \cdot (x_a-x_b)} e^{ip_2 \cdot (x_c-x_d)} \]

\[ \frac{(-1)^2}{(2-2)!} \int_a \int_b \int_c \int_d \frac{1}{2\pi i \hbar} \frac{1}{2\pi i \hbar} \frac{d^2p_1}{(2\pi)^2} \frac{d^2p_2}{(2\pi)^2} \]

\[ n_b(p_1) n_b(p_2) e^{ip_1 \cdot (x_a-x_b)} e^{ip_2 \cdot (x_c-x_d)} \]

\[ \frac{1}{2!} \]

\[ \frac{(-1)^2}{(2x+1)!} \left( \right. \int_a \int_b \int_c \int_d \frac{d^2p_1}{(2\pi)^2} \frac{1}{2\pi i \hbar} n_b(p_1) e^{ip_1 \cdot (x_a-x_b)} \left. \right) ^2 \]

\[ \frac{(2n+1)!}{n!} \left( \int_a \int_b \int_c \int_d \frac{d^2p_1}{(2\pi)^2} \frac{1}{2\pi i \hbar} n_b(p_1) e^{ip_1 \cdot (x_a-x_b)} \right)^n \]

Do sum \[ \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n)!} \right] \]
\[ \text{evaluated shift.} \]
\[ e^{-\frac{1}{2} \int_0^\infty \left( \frac{1}{a} \phi_a - \frac{1}{b} \phi_b \right)^2 dx} \]

\[ = 1 - \frac{1}{2} \int_0^\infty \frac{1}{a} \phi_a - \frac{1}{b} \phi_b \right)^2 dx \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

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\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

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\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]

\[ \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} = \sum_{j=1}^{\infty} \frac{\phi_0}{\phi_j} \sum_{m=1}^{\infty} \frac{\phi_0}{\phi_m} \]