Tunneling in Potentials With Classical Degeneracy

Consider:

\[ V(x) = -\frac{\hbar^2}{2} x^2 + \frac{3}{4} x^4 \]

It has two degenerate minima: \( x_L \) & \( x_R \).

Looking at just one-half of the potential, the G.S. would be \( \psi_L(x) \) & \( \psi_R(x) \).

However, there is a symmetry \( \hat{P} : x \leftrightarrow -x \) , generator \( \hat{\mathcal{P}} \):

\[ \hat{\mathcal{P}} \psi(x) = \psi(-x) \]

e. values \( \pm 1 \), and \( [\hat{\mathcal{H}}, \hat{\mathcal{P}}] \) (commutes with Hamiltonian)

\( E \)-states of \( \hat{\mathcal{H}} \) are also e-states of \( \hat{\mathcal{P}} \)

\[ \psi_S(x) = \frac{1}{\sqrt{2}} (\psi_R(x) + \psi_L(x)) \]

\[ \psi_A(x) = \frac{1}{\sqrt{2}} (\psi_R(x) - \psi_L(x)) \]

These can’t be degenerate: From general arguments of Quantum Mechanics, the G.S. wave function can always be taken to be positive definite.

Tunneling between the two breaks the symmetry:

What is the difference \( EA - E_S = ? \)
Calculate energy splitting. Start with time-indep. Schrödinger eqn.

\[
\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi_A + V(x) \Psi_A = E_A \Psi_A \right) \times \Psi_S
\]

\[
\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi_S + V(x) \Psi_S = E_S \Psi_S \right) \times \Psi_A
\]

\[
\text{multiply by } \Psi_A \text{ & } \Psi_S.
\]

\[
\text{Subtract: (First eqn) - (Second eqn)}
\]

\[
-\frac{\hbar^2}{2m} \left( \frac{d^2}{dx^2} \Psi_S - \frac{d^2}{dx^2} \Psi_A \right) = (E_A - E_S) \Psi_S \Psi_A
\]

\[
-\frac{\hbar^2}{2m} \left( \frac{d}{dx} \Psi_S - \frac{d}{dx} \Psi_A \right) = \frac{1}{2} \Psi_S \Psi_A
\]

Now, integrate over $x$ from $-\infty$ to 0 (region where $\Psi_S$ & $\Psi_A$ de destructively interfere)

\[
-\frac{\hbar^2}{2m} \int_{-\infty}^{0} \frac{d}{dx} \left( \frac{d}{dx} \Psi_S - \frac{d}{dx} \Psi_A \right) \, dx = (E_A - E_S) \int_{-\infty}^{0} \Psi_S \Psi_A \, dx
\]

Use fund. th. of Calc. $\Psi_S(0) - \Psi_A(0)$ vanishes at $-\infty$.

\[
-\frac{1}{2} \text{ half-normalization}
\]

\[
\text{Half-normalization}
\]

Look at sketch:

\[
\Psi_S(0) = 0, \quad \Psi_A(0) = 0 \quad \text{(node)}
\]

\[
\Psi_S'(0) = \frac{1}{\sqrt{2}}, \quad \Psi_A'(0) = \sqrt{2} \Psi_A'(0)
\]

\[
(-2) \times -\frac{\hbar^2}{2m} \left( \frac{d^2}{dx^2} \Psi_L(0) \Psi_S'(0) = E_A - E_S.
\]

\[
+2 \frac{\hbar^2}{2m} \Psi_L(0) \Psi_L'(0) = E_A - E_S.
\]

Use WKB to estimate this (use soln in cl. forbidden reg.)

WKB:

\[
\Psi_L(x) = \frac{C}{|z_{2m}V(x)|^{1/4}} \exp \left[ -\frac{1}{\hbar} \int_{x_L}^{x} \sqrt{2mV(x')} \, dx' \right]
\]

\[
\Psi_L'(x) \sim \exp \left[ -\frac{1}{\hbar} \int_{x_L}^{x} \sqrt{2mV(x')} \, dx' \right]
\]
So,
\[ \Psi_L(0) \Psi''_L(0) \sim \Psi^2_L(0) \sim \exp \left[ \frac{-2}{\hbar} \int_{x_L}^{0} dx \sqrt{2mV(x)} \right] \]

Therefore,
\[ E_A - E_S \sim \frac{2k^2}{m} \exp \left[ \frac{-2}{\hbar} \int_{x_L}^{0} dx \sqrt{2mV(x)} \right] \]

is the splitting.

A more symmetric form can be obtained (needed in generalization to multivariables):

- Use \( \Psi_L(0) = \frac{1}{\sqrt{2}} \left( \Psi_K(0) + \Psi_L(0) \right) \) and \( \Psi''_L(0) = \frac{1}{\sqrt{2}} \left( \Psi''_K(0) - \Psi''_L(0) \right) \)

Then
\[ \left( \Psi_L \frac{d\Psi_L}{dx} \right)_{x=0} = \frac{1}{2} \left( \Psi_K \Psi'_K + \Psi_L \Psi'_L \right) \left( \Psi_K'(0) - \Psi_L'(0) \right) \]

\[ = \frac{1}{2} \left( \Psi_K \Psi'_K - \Psi_L \Psi'_L - \Psi_K \Psi'_L + \Psi_L \Psi'_K \right)_{x=0} \]

since \( \Psi_K(0) = -\Psi_L(0) \) write:
\[ -\Psi_K \Psi'_K + \Psi_L \Psi'_L \]

\[ = \frac{1}{2} \left( 2 \times (\Psi_L \Psi'_K - \Psi_K \Psi'_L) \right)_{x=0} = \Psi_L \frac{d\Psi_K}{dx} - \Psi_K \frac{d\Psi_L}{dx} \]

Then,
\[ E_A - E_S = \left( \frac{1}{2} \right) \times \frac{-k^2}{2m} \left( \Psi_L \frac{d\Psi_K}{dx} - \Psi_K \frac{d\Psi_L}{dx} \right)_{x=0} = \frac{k^2}{m} \left( \Psi_L \frac{d\Psi_K}{dx} - \Psi_K \frac{d\Psi_L}{dx} \right)_{x=0} , \] which can be estimated using WKB.

Therefore, in systems with multiple classical degenerate minima, "quantum fluctuations" will break this degeneracy to give a set of states: the one whose wavefunction is positive definite everywhere is the ground state.
Tunneling in Potentials with Classical Degeneracy (Euclidean action approach)

Start with $S[x]$, and construct the Euclidean action

$$S_E = \int d\tau \left[ \frac{1}{2} m \left( \frac{d^2x}{d\tau^2} \right)^2 + V(x) \right]$$

Extremize the action: $m \frac{d^2x}{d\tau^2} = \frac{dV}{d\tau}$,

but this time there is no bounce solution.

Instead find solution that goes from $x_L(-\infty) \to x_R(+\infty)$,

subject to $x'_L(-\infty) = 0$ and $x'_R(+\infty) = 0$ (by cons. of Eucl. energy).

This solution is the instanton $x^I(\tau)$ (does not return to $x_L$).

Semiclassical tunneling exponential:

$$T \sim e^{-S_E[x^I]/\hbar} = \exp \left( \frac{-1}{\hbar} \int_{-\infty}^{\infty} d\tau \left[ \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x) \right] \right)$$

and by symmetry, $\text{action}(x_L \to 0) = \text{action}(x_R \to 0)$

$$= \exp \left( \frac{-2}{\hbar} \int_{-\infty}^{0} d\tau \left[ \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x) \right] \right)$$
Tunneling in Potentials with Classical Degeneracy (Euclidean Action approach)

Start with action $S[\vec{x}]$, and construct the Euclideanized Action

$$S_E = \int dt \left[ \frac{1}{2} m \left( \frac{d\vec{x}}{dt} \right)^2 + V(\vec{x}) \right]$$

Extremize the action: $m \frac{d^2 \vec{x}}{dt^2} = \frac{dV}{d\vec{x}}$

but this time, there is no bounce solution.

Instead find solution that goes from $x_L(-\infty) \to x_e(+\infty)$,

subject to $x_L'(-\infty) = 0$ and $x_e'(+\infty) = 0$ (by inv. of eucl. energy)

This solution is the Instanton, $x_I(t)$. (does not return to $x_L$).

Semiclassical tunneling exponential:

$$T \sim e^{-S_E[x_I]/\hbar} = \exp \left[ \frac{-1}{\hbar} \int_{-\infty}^{+\infty} dt \left[ \frac{1}{2} m \left( \frac{d\vec{x}_I}{dt} \right)^2 + V(\vec{x}_I) \right] \right]$$

and by symmetry, action ($x_L \to 0$) = action ($0 \to x_e$).

$$= \exp \left[ \frac{-2}{\hbar} \int_{-\infty}^{0} dt \left[ \frac{1}{2} m \left( \frac{d\vec{x}_I}{dt} \right)^2 + V(\vec{x}_I) \right] \right]$$

Explicit calculation for $V(x) = \frac{g}{4} (x^2 - a^2)^2$

Note: Harmonic frequency at bottom of well:

$$m \omega_0^2 = \frac{d^2}{dx^2} V(x=a) = 2g a^2$$

$$\Rightarrow \omega_0 = \sqrt{\frac{g a^2}{m}} \Rightarrow a^2 = \frac{m \omega_0^2}{2g}$$

Maybe better to write: $V(x) = \frac{g}{4} (x^2 - \frac{m \omega_0^2}{2g})$
Search for solutions to \[ m \frac{d^2 x}{d \tau^2} = g \left( \alpha^2 - a^2 \right) x, \]
subject to boundary condition: \( x(-\infty) = -a, \quad x'(-\infty) = 0. \)

Try: \( x_I(t) = a \tanh \left( \gamma (\tau - \tau_0) \right) \)

Solve for \( \gamma \) is arbitrary — time-translation invariance.
\( \tau_0 \) = instanton displacement.

Differential equation becomes:

\[ m \frac{d^2}{d \tau^2} \left( a \tanh (\gamma (\tau - \tau_0)) \right) = g \left( a^2 \tanh^2 (\gamma (\tau - \tau_0)) - a^2 \right) \]

\[ x \tanh (\gamma (\tau - \tau_0)) \]

\[ -2 \frac{m a}{\gamma} \sech^2 (\gamma (\tau - \tau_0)) \tanh (\gamma (\tau - \tau_0)) = -g a^3 \sech^2 (\gamma (\tau - \tau_0)) \tanh (\gamma (\tau - \tau_0)) \]

\[ \Rightarrow \quad \gamma = \sqrt{\frac{g a^2}{2m}} = \frac{\omega_0}{2} \]

Hence \( x_I(t) = a \tanh \left( \frac{\omega_0}{2} (\tau - \tau_0) \right) \)

Hence the semiclassical exponential factor is:

\[ \Gamma \sim e^{-\frac{s_I}{\hbar}} \]

\[ = \exp \left[ -\frac{2}{\hbar} \int_{-\infty}^{\tau_0} \! dt \left[ \frac{1}{2} m \left( \frac{dx_I}{d\tau} \right)^2 + \frac{g}{4} (x_I^2 - a^2)^2 \right] \right] \]

\[ = \exp \left[ -\frac{m^2 \omega_0^2}{2g} \int_{-\infty}^{\tau_0} \! dt \frac{m^2 \omega_0^2}{8g} \sech^4 \left( \frac{1}{2} \omega_0 (\tau - \tau_0) \right) \right] \]

\[ = \exp \left[ -\frac{m^2 \omega_0^2}{3 g} \frac{1}{\hbar} \right], \quad \omega_0 = \frac{8 a^2 \lambda}{m} \text{ is the fundamental frequency at the bottom of well.} \]