Zeta function regularization
\[ \partial^2_E = + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

The (logarithmic) determinant of a second-order elliptic differential operator, \( O \)

is formally divergent (sum of all eigenvalues \( \to \infty \)) — nothing to do with renormalization

This is the reason for evaluating (logarithmic) determinant ratios (of two operators \( O_1 \) & \( O_2 \))

— the formal divergence cancels:

\[ \ln \det O \to \infty \quad \text{but} \quad \ln \frac{\det O_1}{\det O_2} \to \text{finite} \]

In the business of calculating functional determinants, it is sometimes useful to consider the logarithmic determinant of a single operator rather than a ratio.

Thus, we are forced with having to regulate the divergence. One possible regularization scheme is the Zeta function regularization:

Recall:
\[ \ln \det O = \sum_n \ln \lambda_n \] \( \text{(in \ det \ \equiv \ Tr \ln) } \)

Regulate sum by dividing each term by \( \lambda_n^s \) \( (s \text{ \ positive \ number}) \)

\[ \longrightarrow \sum_n \frac{\ln \lambda_n}{\lambda_n^s} \]

\[ = - \frac{d}{ds} \sum_n \frac{1}{\lambda_n^s} \]

\[ = \zeta(s) \quad \text{Zeta function of operator } O \]

\[ = - \frac{d}{ds} \zeta(s) = - \zeta'(s) \]

Then
\[ \ln \det O = \lim_{s \to 0} - \zeta'(s) = - \zeta'(0) \quad \text{shorthand} \]

Note: Zeta functions are tied to a particular operator.

The limit \( s \to 0 \) is formal;

how to ensure convergence at \( s = 0 \)? Analytic continuation
Consider the Riemann zeta function:

\[ \zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]  

convergent if \( \text{Re}(s) > 1 \).

Recall \( \Gamma(s) = \int_0^{\infty} dt \ t^{s-1} e^{-t} \)  

integral rep. of gamma function 

convergent if \( \text{Re}(s) > 0 \).

Multiply/divide \( \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} e^{-t} \) to each term in sum.

\[ \zeta_R(s) = \sum_{n=1}^{\infty} \left[ \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} e^{-t} \right] \frac{1}{n^s} \]

multiply by \( \frac{1}{n^s} \).

\[ = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} e^{-t} \frac{1}{n^s} \]

\[ = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{dt}{n} \left( \frac{t}{n} \right)^{s-1} e^{-t} \]

Rescale integration variable \( t \rightarrow nt \)

\[ = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} e^{-nt} \]

So far our expression is strictly equal to \( \sum_{n=1}^{\infty} \frac{1}{n^s} \), convergent for \( \text{Re}(s) > 1 \), and ill-defined for \( \text{Re}(s) \leq 1 \). But now, pretend sum/integral is convergent for all \( s \), so that we may interchange them.

\[ \longrightarrow \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} \sum_{n=1}^{\infty} e^{-nt} \]

\[ \boxed{\zeta_R(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} \frac{1}{e^t-1}} \]

Standard integral representation of Riemann zeta function.

This will be our new, analytically continued, definition of \( \zeta_R(s) \).

- agrees with \( \sum_{n=1}^{\infty} \frac{1}{n^s} \) for \( \text{Re}(s) > 1 \)
- finite for \( \text{Re}(s) \leq 1 \).
Let's evaluate this at $s=0$ where $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is clearly (very!) divergent.

Write \[ \frac{1}{e^t-1} = \frac{e^{-t/2}}{e^{t/2} - e^{-t/2}} = \frac{e^{-t/2}}{2 \sinh \left( \frac{t}{2} \right)}. \]

\[ \zeta_R(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \; t^{s-1} \frac{e^{-t/2}}{2 \sinh \left( \frac{t}{2} \right)} \]

Note: at $s=0$, integrand is divergent in the lower limit, but $\frac{1}{\Gamma(s=0)} = 0$

So could $\frac{d}{ds}$ be finite?

Subtract small $t$ behavior of integral and add it back.

\[ \frac{1}{\sinh \left( \frac{t}{2} \right)} \approx \frac{1}{t} - \frac{1}{2t^3} + O(t^5) \]

\[ \zeta_R(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \; t^{s-1} e^{-t/2} \left( \frac{1}{2 \sinh \left( \frac{t}{2} \right)} - \frac{1}{t} \right) + \frac{1}{\Gamma(s)} \int_0^\infty dt \; t^{s-1} e^{-t/2} \frac{1}{t} \]

At $s=0$, the divergent small $t$ behavior is removed \Rightarrow finite number

So take $s \to 0$ limit:

\[ \zeta_R(0) = \lim_{s \to 0} \frac{1}{\Gamma(s)} \int_0^\infty dt \; t^{s-1} e^{-t/2} \left( \frac{1}{2 \sinh \left( \frac{t}{2} \right)} - \frac{1}{t} \right) + \lim_{s \to 0} \frac{2^{s-1} \Gamma(s-1)}{\Gamma(s)} \]

\[ = 0 + \left( -\frac{1}{2} \right) \]

\[ \zeta(0) = \frac{-1}{2} \]

Similarly (to be shown)

\[ \zeta'(0) = -\frac{1}{2} \ln 2\pi \]
Write \( \frac{1}{e^t - 1} = \frac{e^{-t/2}}{e^{t/2} - e^{-t/2}} = \frac{e^{-t/2}}{2 \sinh \left( \frac{t}{2} \right)} \), so that

\[ \zeta_R(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \; t^{s-1} e^{-t/2} \left( \frac{1}{2 \sinh \left( \frac{t}{2} \right)} - \frac{1}{t} \right) + \frac{1}{\Gamma(s)} \int_0^\infty dt \; \frac{t^{s-1} e^{-t/2}}{t} \]

At \( s = 0 \), the divergent small \( t \) behavior is removed.

So at \( s = 0 \), \( \frac{1}{\Gamma(s)} \to 0 \), first term vanishes, and second term = \(- \frac{1}{2}\).

So, \[ \zeta_R(0) = -\frac{1}{2} \] - See back for details.

Hurwitz Zeta function (Generalized Zeta function)

\[ \zeta_H(s, z) = \sum_{n=0}^\infty \frac{1}{(n+z)^s} \] special case \( \zeta_H(s, 0) = \zeta(s) \), convergent for \( \text{Re}(s) > 1 \)

Following the same steps as before,

\[ \zeta_H(s, z) = \sum_{n=0}^\infty \frac{2^{s-1}}{\Gamma(s)} \int_0^\infty dt \; \frac{t^{s-1} e^{-t}}{(n+z)^s} \]

\[ = \sum_{n=0}^\infty \frac{2^{s-1}}{\Gamma(s)} \int_0^\infty dt \; \left( \frac{t}{2(n+z)} \right)^{s-1} e^{-t} \]

Rescale integration variable: \( t \to 2(n+z) t \)
\( C_h(s, z) = \sum_{n=0}^{\infty} \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, 2e^{-2(zt+t)} \)

Perform sum over \( n \)

\( = \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, e^{-2zt} \frac{2}{1-e^{-2t}} \)

\( = \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, e^{-2zt} \left(1 + \coth t\right) \)

Can perform integration on first term: \( \rightarrow 2^{-s} z^{-s} \Gamma(s) \)

\( = \frac{z^{-s}}{2} + \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, e^{-2zt} \coth t \)

The integral in the first term is divergent as \( t \to 0 \) like \( \int_0^{\infty} \frac{dt}{t} \sim \ln(t) \), due to \( \coth t \sim \frac{1}{t} \)...

So, add and subtract.

\( = \frac{z^{-s}}{2} + \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, e^{-2zt} \left(\coth t - \frac{1}{t}\right) + \frac{2^{s+1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, e^{-2zt} \frac{1}{t} \)

\( = \frac{z^{-s}}{2} + \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, e^{-2zt} \left(\coth t - \frac{1}{t}\right) + \frac{2^{s+1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, e^{-2zt} \frac{1}{t} \)

\( = \frac{z^{-s}}{2} + \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \, e^{-2zt} \left(\coth t - \frac{1}{t}\right) + \frac{2^{s+1}}{\Gamma(s)} \Gamma(s-1) \frac{1}{z} \)

So, since integral is finite for \( s \to 0 \) and \( \frac{1}{\Gamma(0)} = 0 \), \( 2^{s+1} \) term vanishes at \( s = 0 \).

\( S_m(0, z) = \frac{1}{2} - z \)