Non-topological solitons (e.g. \( Q \)-balls)

Non-topological solitons are stable by virtue of conservation of charge:

\[
L = \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi) + \partial_{\mu} \chi^{x} \partial^{\mu} \chi^{x} - g^2 \phi^2 \chi^{x} \chi^{x}
\]

\[
V(\phi) = \frac{m_{\phi}^2}{2} (\phi - \mu)^2
\]

\( m_{\phi} > 0 \), \( \mu > 0 \), \( \eta > 0 \).

In 1+1 dimensional spacetime:

Invariance under global \( U(1) \):

\( \phi(x) \rightarrow \phi(x) \).

\( \chi(x) \rightarrow e^{i \alpha} \chi(x) \)

\( \delta \chi = i \alpha \chi(x) \quad \delta \chi^{b} = -i \alpha \chi^{b}(x) \)

Noether current:

\[
j_{\mu} = -i \left( \partial_{\mu} \chi^{b} \chi - \partial_{\mu} \chi \chi^{b} \right)
\]

\( Q = \int d\chi \left( -i \chi^{+} \chi + i \chi^{b} \chi^{b} \right) \)

Energy functional

\[
E = \int d\chi \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\partial_{\chi} \chi^{b})^2 + \frac{m_{\phi}^2}{2} (\phi - \mu)^2 + |\chi|^2 + \left| \frac{\partial \chi}{\partial x} \right|^2 + g^2 \phi^2 |\chi|^2 \right]
\]

Trivial minimum: \( \phi = \mu \), \( \chi = 0 \).

Fluctuations: scalar quanta of masses \( g\mu \) and \( m_{\phi} \).
Since charge is conserved, could we have field configurations of a given charge? By virtue of charge conservation, the object would be stable.

- Q-balls.

If charge is non-zero, fields must depend on time: \[ Q = \int dx \left( -i \phi^* \chi + \phi^2 \phi \right) \]

Stationary condition:

\[
\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial^{\mu}}{\partial x^{\mu}} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} = 0
\]

\[
\frac{\partial \mathcal{L}}{\partial x^{\mu}} - \frac{\partial^{\mu}}{\partial x^{\mu}} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} = 0
\]

\[
m^2 (\phi - \phi_0) - \frac{1}{2} \partial_{\mu} \partial^{\mu} \phi = 0.
\]

\[
\phi^2 \phi \chi + \partial_{\mu} \partial^{\mu} \phi = 0.
\]

Ok, to show these are stable, local field configurations, must show energy of non-local fluctuations > local configuration.

Consider small fluctuations of \( \chi(x) \) and \( \phi(x) \) about vacuum.

Linearized theory: \[ E = \int dk \left( a_k^2 \phi_k^* \phi_k + b_k^2 \phi_k^* \phi_k \right) \quad \omega_k = \sqrt{m_x^2 + k^2} \]

\[ Q = \int dk \left( a_k^2 \phi_k^* \phi_k - b_k^2 \phi_k^* \phi_k \right) \]

Consider \( a_k = a \delta(k) \Rightarrow E = m_x (a^2 + b^2) \delta(0) \)

\( b_k = b \delta(k) \Rightarrow Q = (a^2 - b^2) \delta(0) \)

\[ Q + b^2 \delta(0) = a^2 \delta(0) \]

\[ E = m_x Q + 2m_x b^2 \delta(0) \]

\[ E > m_x Q \]