Single-Variable Tunneling (Euclideanized Action)

The semiclassical tunneling exponent can be calculated by evaluating the action in imaginary time — (more rigorously done using path integrals).

Action of one-particle (non-relativistic) system.

\[ S = \int_{t_1}^{t_f} dt \left( \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 - V(x) \right) \]

Trick: go to imaginary time; ch var. \( t \to (i\tau) \)

\[ S = \int_{-i\tau_1}^{-i\tau_f} d\tau \left( \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 - V(x) \right) \]

\[ = \int_{-i\tau_1}^{-i\tau_f} d\tau \left( -\frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x) \right) \]

\[ = i \int_{-i\tau_1}^{-i\tau_f} d\tau \left[ \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x) \right] \]

And now change integration limits to \(+\tau_1 \to +\tau_f\) (Wide Relation) giving the Euclidean Action

\[ iS_E = i \int_{\tau_1}^{\tau_f} \left[ \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x) \right] \]

Now extremize this action (obtain E-L eqn. of mot.)

\[ \frac{\partial L}{\partial x} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}} \right) = 0 \]

\[ L = \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x) \]

\[ \Rightarrow m \frac{d^2 x}{d\tau^2} = \frac{dV}{dx} \]

This equation of motion formally coincides with Newton's eqn, but with the sign of the potential reversed.
To calculate tunneling probability (in semiclassical approx.), follow two steps:

1. Solve eqn. of mot. with particle at \( x_1 \) initially at rest, rolling into the well at right, bouncing at \( x_2 \), and returing to the initial location, \( x_1 \).

This solution is the "bounce" solution, \( x_B(t) \).

First integral of the E-L eqn. of motion gives

\[
\int m \frac{d^2x}{dt^2} \, dx = \int dV \quad \Rightarrow \quad \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 - V(x) = E_{\text{cmv}},
\]

(see facing page) const. of integration.

For a bounce solution, \( KE \) at \( x_1 \) vanishes, and \( V(x_1) = 0 \) \( \Rightarrow \) \( E_{\text{cmv}} = 0 \) (at all times since \( E_{\text{cmv}} \) is conserved).

Hence, the bounce solution satisfies

\[
\frac{1}{2} m \left( \frac{dx_B}{dt} \right)^2 = V(x_B) \quad \Rightarrow \quad \frac{dx_B}{dt} = \sqrt{\frac{2V(x_B)}{m}} \quad \text{or} \quad dt = \sqrt{\frac{m}{2V}} \, dx_B.
\]

Hence, the tunneling semiclassical exponent is \( T \sim e^{-S_E[x_B(t)]} \).

\[
S_E[x_B] = \int_{T_1 = -\infty}^{T_1 = +\infty} \left[ \frac{1}{2} m \left( \frac{dx_B}{dt} \right)^2 + V(x_B(t)) \right] \, dt = \left( \int_{-\infty}^{\infty} m \left( \frac{dx_B}{dt} \right)^2 \, dt \right)
\]

\[
= \int_{-\infty}^{+\infty} \, dt \, 2V(x_B(t)), \quad \text{since action is even in } x_B \text{ and solution even in } T_1,
\]

\[
= 2 \int_{-\infty}^{0} \, d\tau \, 2V(x_B(\tau)) \quad \text{subs. } \tau = \sqrt{\frac{m}{2V}} \, dx_B
\]

\[
= 2 \int_{x_1}^{x_2} dx_B \sqrt{2mV(x_B)}
\]

in agreement with WKB, and matching wavefunctions, for \( E = 0 \).