Sommerfeld-Watson Transformation - Yukawa potential

\[ f(E, q^2) = \sum_{l=0}^{\infty} (2l+1) f_l(E) P_l(\cos \Theta), \quad \cos \Theta = 1 - \frac{q^2}{2k^2} \]

Replace with integral:

\[ f(E, q^2) = \frac{1}{2i} \oint_{c} d \ell \frac{(2l+1) f_l(E) P_l(-\cos \Theta)}{\sin \pi \ell} \]

* \( f_l(E) \) is partial wave amplitude analytically continued to complex \( \ell \).

* \( P_l(z) \) is analytically continued L. poly. \( \equiv \frac{2F_1(-\ell, \ell+1; 1; \frac{1}{2}(1-z))}{\ell} \frac{1}{\ell^2} \frac{1}{(2\ell-1)^2} 4e^z \) (Legendre function) continued to complex \( \ell \). Entire in \( \ell \), analytic in \( z \).

\[ \text{Denominator } \sin(\pi \ell) \text{ generates poles at integer } \ell, \text{ with residue } (-1)\frac{1}{\pi}. \]

Cauchy's residue theorem recovers partial wave sum. CHECK:

\[ f(E, q^2) = \frac{1}{2i} 2\pi i \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell} \sum_{l=0}^{\infty} (2l+1) f_l(E) P_l(-\cos \Theta) \]

\[ = \sum_{l=0}^{\infty} (2l+1) f_l(E) P_l(\cos \Theta) \quad \checkmark \]

Distort contour of integration so

so that it runs down \( \text{Re } \ell = -\frac{1}{2} \) line.

For Yuk. pot, \( |f_l(E)| \leq O(E) \frac{e^{-\gamma(E)}}{\ell!} \), vanishes as \( \text{Re } \ell \to \infty \) \Rightarrow \text{quarter-circle contribution vanishes.}

What about small sectors in left-half plane? Also, \( P_l \) has good behavior as \( \ell \to \infty \).

After picking up Regge poles,

\[ f(E, q^2) = \frac{1}{2i} \int_{-i\infty}^{i\infty} d \ell \frac{(2l+1) f_l(E) P_l(-\cos \Theta)}{\sin \pi \ell} + \frac{1}{2i} 2\pi i \sum_{\ell=0}^{\infty} \text{Res}_{\ell} f_l(E) P_{\ell+1}(\cos \Theta) \]

rewrite differently.
Sommerfeld representation

\[ f(E, q^2) = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{d\lambda}{\sin(\pi \lambda)} \left( 2\lambda + 1 \right) \frac{P_\lambda(E)}{P_\lambda(-\cos \theta)} + \sum_n \frac{(2\lambda_n E + 1) \beta_n(E) P_{\lambda_n}(E)}{\sin \pi \lambda_n(E)} \]

Background integral

Regge-pole part

Has two important uses:

1. Analyticity in \( q^2 \):

Regge-Pole part:

\( P_\lambda(z) \) analytic for all \( z \), except for cut \( -\infty < z \leq -2 \)

\( \Rightarrow \) analytic in \( \cos \theta \), except for cut \( 1 \leq \cos \theta < \infty \)

\[ \Rightarrow q^2, \text{ cut } -\infty < q^2 \leq 0 \]

Background integral:

\[ \frac{P_{\lambda}(z)}{\sin(\pi \lambda)} \leq \rho(\theta) E^{-\frac{1}{2}} e^{-|\arg E|} \rightarrow \text{integral converges for all } \theta. \]

Thus, \( f(E, q^2) \) has only a left-hand cut starting at \( q^2 = 0 \).

BUT: together with large Lehmann ellipse, left-hand cut is shortened.

\[ \Rightarrow \text{(can finally) proceed to obtain dispersion relation in } q^2, \Rightarrow \text{Mandelstam rep.} \]

2. Large \( q^2 \) behavior: (from Regge theory)

For large \( \lambda \) (see facing page)

\[ q^2 = 2\lambda^2 (1 - \cos \theta) \]

\[ P_\lambda(\lambda) \rightarrow \text{const } \lambda^L \]

So, background integral along line \( \lambda = -\frac{1}{2} \), decays as \( |\cos \theta|^{-\frac{1}{2}} \rightarrow 0. \)

\[ \int \text{cont. on next page.} \]
But each term in the Regge pole part goes like \( \sim (\cos \theta)^{\alpha_3(E)} (q^2)^{\alpha_3(E)} \) as \( q^2 \to \infty \). Therefore, the growth of \( f(E, q^2) \) as \( q^2 \to \infty \) is dominated by the Regge pole furthest to the right (in \( l \)-plane):

\[
f(E, q^2) \xrightarrow{q^2 \to \infty} \text{(const.)} (q^2)^{\alpha_{\text{max}}(E)} \quad \quad \text{[Regge expansion]}
\]

This has important implications for writing down dispersion relations in \( q^2 \).
If the (Yukawa) potential has bound state with maximum angular momentum \( l_{\text{max}} \), then there exists at least one trajectory with \( \text{Re} \alpha(E) \geq l_{\text{max}} \). Therefore, \( f(E, q^2) \) behaves like \( (q^2)^l \) at least. Thus, we must write a dispersion relation for

\[
f(E, q^2) = \frac{f(E, q^2)}{(q^2)^m} \quad \text{m integer } \geq l_{\text{max}},
\]

that is - a dispersion relation with \( m \) subtractions.

Another important application: relativistic scattering theory.

\( E^2 - q^2 \) are Mandelstam \( s, t \)

\[
\Rightarrow \quad \text{So, if for } A + B \to C + D, \quad f(s, t) \xrightarrow{t \to \infty} \text{const.} t^{\alpha_{\text{max}}(s)}
\]

can expect \( A + \bar{C} \to \bar{B} + D, \quad f(s, t) \xrightarrow{s \to \infty} \text{const.} s^{\alpha_{\text{max}}(t)}, \)

by crossing symmetry.

Large exchanged momentum behaviour of scattering amplitude for one process determines large C.O.M energy behaviour of amplitude for related process.

* In 1959, Mandelstam completes the Regge expansion (a series in \( q^2 \to \infty \)) by proving the analytic behaviour of \( f_s(E) \) for \( \text{Re}[s] < -\frac{1}{2} \). [Yukawa potential]