Center of Mass frame Kinematics

2 → 2 Elastic Scattering

\[ \begin{align*}
E_1 & \rightarrow E'_1 \bar{p}'_1 \\
E_2 & \rightarrow E'_2 \bar{p}'_2 \\
\text{mass} & = m_1 \\
\bar{p}'_1 & = (0, 0, 1) |\vec{p}| \\
\bar{p}'_2 & = (\sin \Theta_{cm}, 0, \cos \Theta_{cm}) |\vec{p}| \\
\text{mass} & = m_2 \\
\bar{p}'_2 & = (-\sin \Theta_{cm}, 0, -\cos \Theta_{cm}) |\vec{p}| \\
\text{com energy:} & \quad E_{cm} = E_1 + E_2 \\
\text{reduced mass:} & \quad \mu = \frac{m_1 m_2}{m_1 + m_2}
\end{align*} \]

COM frame defined by \( \bar{p}'_3 = -\bar{p}'_2 \Rightarrow |\bar{p}'_2| = |\bar{p}'_3| \)

Conservation of momentum:
\[ \bar{p}'_3 + \bar{p}'_2 = \bar{p}'_2 + \bar{p}'_3 \Rightarrow \bar{p}'_3 = -\bar{p}'_2 \Rightarrow |\bar{p}'_3| = |\bar{p}'_2| \]

Conservation of energy:
\[ E_1 + E_2 = E'_1 + E'_2 \]
\[ \frac{1}{2} \bar{p}'_1^2 + \frac{1}{2} \bar{p}'_2^2 = \frac{1}{2} \bar{p}'_1^2 + \frac{1}{2} \bar{p}'_2^2 \Rightarrow \frac{1}{2} |\bar{p}'_1|^2 + \frac{1}{2} |\bar{p}'_2|^2 = |\bar{p}'_1|^2 + |\bar{p}'_2|^2 \]

Magnitudes of all four momenta the same \( \Rightarrow E_1 = E_1' \quad E_2 = E_2' \).

Response variable \( \Theta_{cm} \) is no longer tied to \( E_1' \) or \( E_2' \).

Momentum transfer:
\[ q' = \bar{p}'_1' - \bar{p}'_2' = -(\bar{p}'_2' - \bar{p}'_1') \]
\[ q'^2 = \bar{p}'_1'^2 + \bar{p}'_2'^2 - 2 \bar{p}'_1' \bar{p}'_2' \]
\[ = 2|\bar{p}'_1|^2 - 2|\bar{p}'_1|^2 \cos \Theta \]
\[ = 2|\bar{p}'_1|^2 (1 - \cos \Theta) \]
\[ = 4|\bar{p}'_1|^2 \sin^2(\Theta/2) \] or \[ = 8\mu (E_1 + E_2) \sin^2(\Theta/2) \]