Classical scattering

There are three "paths" described here:

actual orbit: satisfies \( m \ddot{\mathbf{x}}(t) = -\nabla V(\mathbf{x}) \)

\[
\begin{align*}
\text{in asymptote: } \mathbf{x}_\text{in}(t) &= \mathbf{z}_\text{in} + \mathbf{v}_\text{in} t \\
\text{out asymptote: } \mathbf{x}_\text{out}(t) &= \mathbf{z}_\text{out} + \mathbf{v}_\text{out} t \\
\end{align*}
\]

satisfy:

\[
\begin{align*}
\mathbf{x}(t) &\overset{t \to -\infty}{\longrightarrow} \mathbf{x}_\text{in}(t) \\
\mathbf{x}(t) &\overset{t \to +\infty}{\longrightarrow} \mathbf{x}_\text{out}(t) \\
\end{align*}
\]

In classical scattering, a scattering event is completely characterized once the asymptotes are given.

Furthermore, the correspondence between \( \mathbf{x}_\text{out}(t) \) and \( \mathbf{x}_\text{in}(t) \) is given by the actual orbit.

Quantum scattering

Same story.

Actual orbit satisfies \( i\hbar \frac{d}{dt} \psi(t) = \hat{H} \psi(t) \)

\[
\hat{H} = \frac{-\hbar^2}{2m} \nabla^2 + V(x)
\]

then In asymptote: satisfies \( i\hbar \frac{d}{dt} \psi_\text{in}(t) = \hat{H}_0 \psi_\text{in}(t) \)

Out asymptote: satisfies \( i\hbar \frac{d}{dt} \psi_\text{out}(t) = \hat{H}_0 \psi_\text{out}(t) \)

\[
\begin{align*}
|\psi(t)\rangle &= \hat{U}(t) |\psi_\text{in}(t)\rangle \quad t \to -\infty \quad \hat{U}_0(t) |\psi_\text{in}(t)\rangle = |\psi_\text{in}(t)\rangle \\
|\psi(t)\rangle &= \hat{U}(t) |\psi_\text{out}(t)\rangle \quad t \to +\infty \quad \hat{U}_0(t) |\psi_\text{out}(t)\rangle = |\psi_\text{out}(t)\rangle \\
\end{align*}
\]

In both cases (classical and quantum), not all orbits \( \psi(t) \) will tend to "free" in/out states.

- Bound states for strong enough potentials (e.g., planetary orbits, or atomic orbitals)
- For Coulomb, states do not approach asymptote.
Asymptotic condition:

For every $|\Psi_{in}\rangle$ in $\mathcal{H}$, there is a corresponding $|\Psi\rangle$ such that

$$U(t)|\Psi\rangle = U_0(t)|\Psi_{in}\rangle \xrightarrow{t \to \infty} 0.$$  

Similarly, for every $|\Psi_{out}\rangle$ in $\mathcal{H}$, there is a $|\Psi\rangle$ such that:

$$U(t)|\Psi\rangle = U_0(t)|\Psi_{out}\rangle \xrightarrow{t \to \infty} 0.$$  

Basically, $|\Psi(t)\rangle$ becomes physically indistinguishable from $|\Psi_{in}(t)\rangle$ in far future and $|\Psi(t)\rangle$ from $|\Psi_{out}(t)\rangle$ in far past.

Møller Wave Operators:

Multiply both sides of (v) & (xx) by $U(t)$, and solve for $|\Psi\rangle$

$$|\Psi\rangle = \lim_{t \to \infty} U(t)U_0(t)|\Psi_{in}\rangle = \Omega_+|\Psi_{in}\rangle$$  

$$|\Psi\rangle = \lim_{t \to \infty} U^+(t)U_0(t)|\Psi_{out}\rangle = \Omega_-|\Psi_{out}\rangle$$

Møller wave operators act on free (in and out) states to give back fully interacting state.

$$\Omega_+\Omega_+ = 1 \hspace{1cm} \text{but} \hspace{1cm} \Omega_+\Omega_- \neq 1.$$  

Møller wave operators are only isometrics, not unitary.
Scattering operator

\[ |\psi_i\rangle \to \frac{1}{t-\infty} (t) |\psi_f\rangle \& \to \omega \cdot |\psi_f\rangle \]

Also used to write down increase relation.

\[ \omega \downarrow \]

\[ \omega \uparrow \]

Application: To calculate exponentially relevant scattering probabilities.

\[ I \psi_{in} \to e^{-\omega t} U(\omega t) I \psi_f \to \omega \cdot I \psi_f \]

\[ \text{Nearestby state (defined at } t=0) \]

\[ I \psi_{in} \to e^{\omega t} U(\omega t) I \psi_f \to 0 \cdot I \psi_f \]

\[ \text{Nearestby state.} \]

MAIN SCATTERING STORY-LINES:

apply \( \mathbb{S}^+ \)

apply \( \mathbb{S}^- \)

or even calculate overlap here

But might as well conserve overlap here

Asymptotic overlap.
I am interested in the overlap

\[ \omega(\psi' \otimes \psi) = \left| \left\langle \psi'_{\text{out}} | \psi_{\text{in}} \right\rangle \right|^2 \]

use

\[ \left\langle \psi'_{\text{out}} \right| = \left\langle \psi'_{\text{out}} | \Omega \right\rangle \]

and

\[ | \psi \rangle_{\text{out}} = | \Omega \rangle | \psi_{\text{in}} \rangle \]

\[ = \left| \left\langle \psi'_{\text{out}} | \Omega \right\rangle \Omega \right\rangle \left| \psi_{\text{in}} \right\rangle \]

define:
Scattering operator \( S \)

\[ = \left| \left\langle \psi'_{\text{out}} | S | \psi_{\text{in}} \right\rangle \right|^2 \]

applying to
For a trial scattering experiment \( (\alpha, \omega) \) will result

\[ \omega(\alpha, \omega, \psi_{\text{in}}) = \left| \left\langle \alpha, \omega, \psi_{\text{in}} \right| S \right| \psi_{\text{in}} \rangle \right|^2 \]

\[ \check{\text{a state}} \quad \text{probability in } \mathbb{C}^2 \]
\[ \text{with differential angular width} \]

\[ \left( \frac{d \Omega}{d \Omega} \right) \]

\[ \text{Correction to QM notes:} \]

\[ \text{originally, had} \quad \left\langle \psi_{f}(\alpha, \omega, t_i) | \psi_f(\alpha, \omega, t_f) \right\rangle = \left\langle \psi_f | U(\alpha, \omega, t_f, t_i) | \psi_f \right\rangle \]

\[ = \left\langle \psi_f | U_\omega(t_f) U_\alpha(t_i, t_f) | \psi_{\text{out}} \right\rangle \]

\[ = \left\langle \psi_f | \exp \left( i \frac{\hbar}{\mu} \frac{t_f - t_i}{\hbar} \right) \right\rangle \]

\[ \text{since the high energy states are in the mono-chromatic limit} \]

\[ \text{and in the} \]
Scattering operator

Finally, we are now able to disperse with the exponentially uninteresting actual orbit, and construct a map from in asymptote to out asymptote.

\[ |\Psi_{in}(t)\rangle = \hat{\Omega}_{-}^\dagger |\Psi(t)\rangle = \Omega_{-}^\dagger \Omega_{+}^\dagger |\Psi_{in}(t)\rangle = \mathcal{S}^\dagger |\Psi_{in}(\infty)\rangle \]

Define:

Scattering operator \( \mathcal{S}^\dagger |\Psi_{in}(\infty)\rangle = \Omega_{-}^\dagger \Omega_{+}^\dagger |\Psi_{out}(-\infty)\rangle \)

Act of \( \mathcal{S} \) on \( |\Psi_{in}\rangle \) in pictures

\[ \mathcal{S} = \lim_{t \to \infty} U_0^\dagger(t) U(t) \]

In these three frames, the standard time-dependent scattering event occurs.

The scattering operator takes \( |\Psi_{in}\rangle \) and outputs \( \mathcal{S} |\Psi_{out}\rangle \)

(Effectively like the infinite, the limit of interacting picture time-ev. operator)