Formal properties of Müller and Scattering operators.

Intertwining Relation for Müller operators

\[ \hat{H} \Omega_{\pm} = \hat{\Omega}_{\pm} \hat{H}_0 \]

Proof: Consider \( e^{i \hat{H} t} \Omega_{\pm} \)

\[
e^{i \hat{H} t} \Omega_{\pm} = e^{i \hat{H} t} \left[ \lim_{t \to \infty} e^{i \hat{H}_0 t} e^{-i \hat{H}_0 t} \right]
\]

\[
= \lim_{t \to \infty} \left[ e^{i \hat{H}(t+2)} e^{-i \hat{H}_0 t} \right]
\]

\[
= \lim_{t \to \infty} \left[ e^{i \hat{H}(t+2)} e^{-i \hat{H}_0 (t+2)} \right] e^{i \hat{H}_0 t}
\]

shift since is CRITICAL! \( \Omega_{\pm} e^{i \hat{H}_0 t} \)

Now differentiate with respect to \( \tau \), and put \( \tau = 0 \):

\[
\left( i \hat{H} e^{i \hat{H} t} \right) \Omega_{\pm} = \Omega_{\pm} \left( i \hat{H}_0 e^{i \hat{H}_0 t} \right)
\]

\[ \hat{H} \Omega_{\pm} = \Omega_{\pm} \hat{H}_0 \]

Important consequence:

If \( |\psi_{in}\rangle \in \mathcal{H}_0 \) is an e-state of \( \hat{H}_0 \) with energy \( E \),

then \( |\psi\rangle = \Omega_{\pm} |\psi_{in}\rangle \in \mathcal{H} \) is an e-state of \( \hat{H} \) with same energy \( E \):

\[
\hat{H} |\psi\rangle = \hat{H} \Omega_{\pm} |\psi_{in}\rangle = \Omega_{\pm} \hat{H}_0 |\psi_{in}\rangle = E \left( \Omega_{\pm} |\psi_{in}\rangle \right) = E |\psi\rangle
\]

Therefore, the initial state of energy \( E \) evolves to an actual orbit scattering state with the same energy.
Orthogonality theorem

Bound states are orthogonal to scattering states: \( B \perp R \)

Asymptotic completeness:

\[
\begin{align*}
\{ \text{all states} \} &= \{ \text{all states} \} \\
&= \{ \text{all states orthogonal to the bound states} \}
\end{align*}
\]

\( R^+ \quad R^- \)

Space of asmp. IN states \( \widetilde{H}_0 \) space of actual orbits \( \tilde{H} \) space of asmp. out states \( \tilde{H}_0 \)

Spanned by eigenstate of: \( \tilde{H}_0 \)\( \tilde{H} \)\( \tilde{H}_0 \)

Symbol: \( \Psi_{\text{asmp}} = \mathcal{S} \quad \mathcal{H} = R \oplus B \quad \Psi_{\text{asmp}} = \mathcal{S} \)

\( |\psi_{\text{in}}\rangle \xrightarrow{\Omega^-} R_- \quad \Omega_+ \quad R^+ \xrightarrow{\Omega^-} |\psi_{\text{out}}\rangle \)

\( \Psi_{\text{out}} = \mathcal{S} |\psi_{\text{in}}\rangle \)

with \( |\psi_{\text{out}}\rangle, |\psi_{\text{in}}\rangle \in \mathcal{S} \)

**Assumptions:**

I. \( \mathcal{V}(r) = O\left( \frac{1}{r^3 + \epsilon} \right) \) as \( r \to \infty \)

II. \( \mathcal{V}(r) = O\left( \frac{1}{r^{2\alpha} - \epsilon} \right) \) as \( r \to 0 \)

III. \( \mathcal{V}(r) \) is continuous, except for finite number of discontinuities.

**Note!!** Space of in-states and out-states are the same space!