Operator formulation of non-relativistic scattering theory.

S-matrix element:
Transition probability of asymptotic in-state to asymptotic out-state:

\[ W(\text{out} \leftarrow \text{in}) = \langle \text{out} | \hat{S} | \text{in} \rangle \]

Properties of problem can give us form for \( \hat{S} \):
- Conservation of energy: \( [\hat{H}_0, \hat{S}] = 0 \)
  
\[ \Rightarrow \langle \mathbf{E}' | \hat{S} | \mathbf{E} \rangle = (2\pi) \delta(E' - E) \left( \text{shell} \right) \]

The \( T \)-matrix element:
For short range potentials, we can expect: \( \hat{S} = \hat{1} + i \hat{T} \leftarrow \hat{\mathbf{p}} \)

\[ \langle \mathbf{E}' | \hat{S} | \mathbf{E} \rangle = \langle \mathbf{E}' | \hat{1} + i \hat{T} | \mathbf{E} \rangle \]

\[ = \langle \mathbf{E}' | \hat{E} \rangle + i \langle \mathbf{E}' | \hat{T} | \mathbf{E} \rangle \]

\[ = \delta^{(3)}(\mathbf{E}' - \mathbf{E}) + 2\pi i \delta(E' - E) \left( \text{shell} \right) \]

Define scattering amplitude by:

\[ f(\mathbf{E}' \leftarrow \mathbf{E}) = \frac{-i(2\pi)^2 m}{t(\mathbf{E}' \leftarrow \mathbf{E})} \]

so that

\[ \langle \mathbf{E}' | \hat{S} | \mathbf{E} \rangle = \delta^{(3)}(\mathbf{E}' - \mathbf{E}) - 2\pi i \delta(E' - E) \left( \frac{\hbar^2}{(2\pi)^2 m} f(\mathbf{E}' \leftarrow \mathbf{E}) \right) \]

Reason for making all these definitions:
- Elements of \( \hat{S} \) and \( \hat{T} \) are highly singular functions of \( \mathbf{E}' \) & \( \mathbf{E} \) due to \( \delta(E' - E) \).
- However, \( f(\mathbf{E}' \leftarrow \mathbf{E}) \) is well-behaved and is mathematically more manageable.