Quantum Cross Section

For $s_{in}$, consider "single-event experiment."

\[ \tilde{\psi}_{in}(E') = \langle E' | \psi_{in}(E') \rangle \]

Incoming wave packet:
\[ \text{Scattering center} \]
\[ \text{outgoing wave} \]
\[ \tilde{\psi}_{out}(E') = \langle E' | \psi_{out} \rangle \]

What is the probability of finding the scattered particle in solid angle element $d\Omega$? (not interested in measuring magnitude $|E'|$, so integrate over it)

\[ W(d\Omega \leftarrow \tilde{\psi}_{in}) = d\Omega, \int_0^{\infty} dk' k'^2 |\tilde{\psi}_{out}(k')|^2 \]

where:
\[ \tilde{\psi}_{out}(k') = \int d^3k' \langle E' | \tilde{S} | k' \rangle \tilde{\psi}_{in}(k') \]

This setup is useless because precise shape of $\tilde{\psi}_{in}(k')$ is never known.

Instead, run this "single-event experiment" over and over again.

Don't expect exact same wavepacket to be produced each time, so "averaging" over all events would give a final answer depending only on mean incoming wavenumber $K_0$ (c.f., central limit theorem).

Model in a phenomenological way:

The wavepackets differ only by random lateral displacements $b$,

So that $\tilde{\psi}_{in}(E, b) = \tilde{\psi}_{in}(E) e^{-iE \cdot b}$

$b = \text{random variable}$

(target)

(very analogous to impact parameter of classical collision theory).
Then, number of particles scattered \( N \) into solid angle \( d\Omega \) over the course of the entire experiment is:

\[
N(\Delta\Omega) = \sum_i w(d\Omega_i) \Psi_{in}^*(E_i, b_i) \quad i = \text{labels individual events.}
\]

For large number of incoming particles, can write sum as an integral over the "sheet of particles" uniform density \( n_{inc} \) (particles per):

\[
N(\Delta\Omega) = n_{inc} \int_{\text{Beam profile}} d^2b \quad w(d\Omega_i) \Psi_{in}^*(E_i, b_i)
\]

Assumption:

Beam size is much larger than size of target.

Or mathematically, domain of integration is much larger than range of potential.

### Safe to extend domain of integral over whole region.

(Extension doesn't contribute to non-forward scattering)

\[
N(\Delta\Omega) = n_{inc} \int d^2b \quad w(d\Omega_i) \Psi_{in}^*(E_i, b_i)
\]

Compare with classical def. of cross section, and define this as quantum cross section.

\[
\sigma(d\Omega_i) \quad \Psi_{in}(E_i)
\]

Assumption fails for long-range potentials like (Coulomb). Task now is to compute \( \sigma(d\Omega_i) \) in terms of \( f(E_i \rightarrow E) \) cross section scattering amplitude.
Computation of quantum cross section:

\[ \sigma(d\Omega, \rightarrow \tilde{\Psi}_m^i(E)) = \int d^2 \tilde{b} \quad w \left( d\Omega, \rightarrow \tilde{\Psi}_m^i(E, \tilde{b}) \right) \]

insert expression from single-event experiments

\[ = \int d^2 \tilde{b} \quad d\Omega \int_0^\infty dk' k'^{12} \left| \tilde{\Psi}_m^i(E', \tilde{b}') \right|^2 \]

\[ = \int d^2 \tilde{b} \quad d\Omega \int_0^\infty dk' k'^{12} \left| \int d^3 k' \left< E' \right| \tilde{\xi} \left| E \right> \tilde{\Psi}_m^i(E) \left( E', \tilde{b}' \right) \right|^2 \]

but \( \left< E' \right| \tilde{\xi} \left| E \right> = \delta^{(3)}(E' - E) + \frac{i \hbar^2}{2\pi m} \delta(E_{E'} - E) f(E' \leftrightarrow E) \)

**FORWARD** \quad **NON-FORWARD**

Restriction: Don't look in forward region (down beam pipe). Then,

\[ \sigma(d\Omega, \rightarrow \tilde{\Psi}_m^i(E)) = \int d^2 \tilde{b} \quad d\Omega \int_0^\infty dk' k'^{12} \left| \int d^3 k \quad \frac{i \hbar^2}{2\pi m} \delta(E_{E'} - E) f(E' \leftrightarrow E) \tilde{\Psi}_m^i(E) \left( E', \tilde{b}' \right) \right|^2 \]

\[ = \frac{d\Omega}{(2\pi)^2 m^2} \int d^2 \tilde{b} \int_0^\infty dk' k'^{12} \left[ \int d^3 k \quad \delta(E_{E'} - E) f(E' \leftrightarrow E) \tilde{\Psi}_m^i(E) e^{-iE' \cdot \tilde{b}} \right] \]

\[ \times \left[ \int d^3 k'' \delta(E_{E''} - E) f(E' \leftrightarrow E'') \tilde{\Psi}_m^i(E') e^{iE'' \cdot \tilde{b}} \right] \]

1. Integrate over \( d^2 \tilde{b} \):

Write \( \tilde{E} = \tilde{E}_\perp + \tilde{E}_\parallel \) (where \( \tilde{E}_\parallel \cdot \tilde{b} = 0 \))

\[ s_0, \quad \int d^2 \tilde{b} \quad e^{-i(\tilde{E} - \tilde{E}''), \tilde{b}'} = (2m)^2 \delta^{(2)}(\tilde{E} - \tilde{E}'') \]

2. Then, using \( \delta(E_{E'} - E) \), put:

\[ \delta(E_{E'} - E_{E''}) \rightarrow \delta(E_E - E_{E''}) = \frac{2m}{\hbar^2} \delta(E_{E}^2 - E_{E''}^2) \]

\[ = \frac{2m}{\hbar^2} \delta(\tilde{E}_\perp^2 - \tilde{E}_{E''}\perp^2) \]

\[ = \frac{2m}{\hbar^2} \delta \left( \tilde{E}_\perp^2 + \tilde{E}_\parallel^2 - \tilde{E}_{E''}\perp^2 - \tilde{E}_{E''}\parallel^2 \right) \]

\[ \rightarrow \frac{2m}{\hbar^2} \delta \left( k_\perp^2 - k_{E''}\perp^2 \right) \]

\[ = \frac{m}{\hbar^2 k_\perp} \delta(k_\perp - k_{E''}\perp) \]
So that \( \odot \) \& \( \ominus \) together gives:

\[
(2\pi)^2 (E'_L - E''_L) \delta(E_{k'} - E_{k''}) = \frac{(2\pi)^2 m}{\hbar^2 \kappa_n} \delta(\kappa - \kappa'')
\]

3) Integrate over \( d^3k'' \), fixing \( \kappa'' \rightarrow \kappa \)

\[
\sigma (d\Omega \leftrightarrow k_0) = \frac{d\Omega}{(2\pi)^2 m^2} \int_0^{\infty}dk' k'^2 \int_0^{\infty}dk \frac{1}{k_n} \delta(E_{k'} - E_k) \left| f(k' \leftrightarrow k) \tilde{\psi}_{in}(k) \right|^2
\]

\[
\times \frac{(2\pi)^2 m}{\hbar^2 \kappa_n} \delta(k' - k)
\]

Integrate over \( k' \), fixing \( k' \rightarrow k \)

\[
= \frac{d\Omega}{m^2} \frac{\hbar^2}{2} \int d^3k \frac{m}{\hbar^2 \kappa_n} \left| f(k' \leftrightarrow k') \tilde{\psi}_{in}(k) \right|^2
\]

Now, investigate functional forms of

- probability amplitude \( \left| f(k' \leftrightarrow k) \right|^2 \)
- and wavepacket \( \left| \tilde{\psi}_{in}(k) \right|^2 \)
Assumption:

If \( \left| f(k_\perp - k') \right|^2 \) varies sufficiently slowly over the region that \( \left| \vec{E}_{in}(k) \right|^2 \) has support, then one can make approx:

\[
\frac{k}{k_\parallel} f(k_\perp - k') \approx \frac{k_0}{k_0} f(k_\perp - k_0) + \text{(small)}
\]

Then:

\[
\sigma (d\omega - k_\omega) = d\omega \left| f(k_\perp - k_\omega) \right|^2 \int d^2k | \vec{E}_{in}(k') |^2 = 1 \quad \text{(Normalized)}
\]

\[
\frac{d\sigma}{d\omega} = \left| f(k_\perp - k') \right|^2
\]

Failure of assumption:

- At a resonance, \( f(k_\perp - k') \) is rapidly varying.
  
  If \( |\vec{E}_{in}(k)| \) is not narrow enough, resonant line will be broadened.

- Screened long range potentials generate highly oscillatory amplitudes (essentially an extreme form of diffraction)

\[
f(k)
\]

(almost always invisible to experimentalists)