Effective Range Approximation for low energy scattering.

Start with

$$\cot \delta_\alpha (k) = \frac{k n^2 (kx) - \gamma j_1 (kx)}{k c j_1 (kx) - \gamma j_1 (kx)} \bigg|_{x=\alpha}$$

we found series expansion in $k$ ($k=0$) is

$$\cot \delta_\alpha (k) = -\frac{1}{ka} \frac{1 + \alpha \gamma_0}{\alpha \gamma_0} + \mathcal{O}(k)$$

meaning by $\alpha$:

$$k \cot \delta_\alpha (k) \equiv -\frac{1}{\alpha} \frac{1 + \alpha \gamma_0}{\alpha \gamma_0} + \ldots$$

$$= -\frac{1}{\alpha_0} \quad \text{where } \alpha_0 \equiv \text{scattering length}.$$  

This expression is true of scattering of square well (in 3 dimensions).

This expression is approximately true for other potentials at very low energies

$$k \cot \delta_\alpha (k) \approx \frac{1}{\alpha_0} + \frac{1}{2} \frac{b_0}{\alpha_0} k^2 + \ldots$$

(can be shown in general)

Recall

$$f \equiv \frac{1}{2ik} \left( e^{2i\delta_\alpha (k)} - 1 \right) = \frac{1}{2ik} \cot \delta_\alpha (k) - i$$

Use effective range approx

$$= -\frac{1}{2ik} \left( \frac{1}{\alpha_0} + i \right)^{-1} + \ldots$$

Which has pole at $\alpha = i/\alpha_0$.

$$\Rightarrow$$ corresponds to bound state of energy

$$E_b = \frac{\hbar^2}{2m} \left( \alpha_0 \right)^2 = -\frac{\hbar^2}{2m} \alpha_0$$

But bound state wavefunction has form

$$\psi \sim e^{2\pi i \frac{E_b}{\hbar^2} x} = e^{-\frac{1}{\alpha_0} x}$$

$$\Rightarrow$$ Scattering length $\approx$ extension of bound state.
How does scattering length change with potential depth?

- Keep scattering energy fixed, $\epsilon \equiv \text{const}$.

$$a_0 = \lim_{k \to 0} \frac{\epsilon}{k} \tan \frac{\pi}{2}$$

As potential depth increases,

- $s$-wave phase shift approaches $\pi/2$, and upon crossing bound state threshold, heads towards $\pi$.

Then, scattering length is:

- negative before BS threshold
- positive after BS threshold

If scattering length is large and:

- negative $\Rightarrow$ potential on verge of supporting bound state
- positive $\Rightarrow$ potential barely supporting bound state (weakly bound)

Experimentally measuring scattering length:

$$\frac{d\sigma}{d\Omega} = |f_0(\theta, \phi)|^2 = |\sum_{i=0}^\infty (2l+1) f_l(k) P_l(\cos \theta)|^2$$

But only $l=0$ partial wave contributes at low energy:

$$\frac{d\sigma}{d\Omega} = |f_{l=0}(k)|^2 = \left| \frac{1}{2ik} \frac{2i-1}{k a_0} \right|^2 = \frac{a_0^2}{1+k a_0^2}$$

(csr of $a_0$ undetermined)

As $\epsilon \to 0$:

$$\sigma(\epsilon=0) = 4\pi a_0^2$$

(4 times classical result)