Non-existence of partial wave series for Coulomb amplitude

The scattering function $f_{2,1}^{\text{coul}}(k)$ can be projected onto $y_{2,1}^{\text{coul}}(k)$, which can also be summed to recover $y_{2,1}^{\text{coul}}(k)$:

$$y_{2,1}^{\text{coul}}(k) = \sum_{\ell=0}^{\infty} (2\ell+1) R_{\ell}^{(2)} k^\ell P_{\ell}(\cos \Theta) \quad \leftrightarrow \quad R_\ell(k) = \frac{1}{2} \int_0^\pi \sin(\theta) y_{\ell,2,1}^{\text{coul}}(k) P_{\ell}(\cos \Theta) d\theta.$$ 

The same does not hold for scattering amplitudes $f_{2,1}^{\text{coul}}(k')$ and $f_{2}^{\text{coul}}(k)$.

1) Divergence of the Coulomb partial wave series:

$$f_{2,1}^{\text{coul}}(k) = \sum_{\ell=0}^{\infty} (2\ell+1) f_{2,1}^{\text{coul}}(k) P_{\ell}(\cos \Theta)$$

Investigate large $\ell$ behavior of summand:

$$f_{2,1}^{\text{coul}}(k) = \frac{1}{2i\kappa} \left( e^{2i\alpha(k)} - 1 \right)$$

where

$$\sigma_{2}(k) = \text{Arg} \Gamma(1 + i\gamma(k))$$

$$= \text{Arg} \left( i + i2 \right) \Gamma(1 + i2)$$

$$= \text{Arg} \left( 1 + i2 \right) + \text{Arg} \Gamma(1 + i2)$$

$$\sigma_{2}(k) = \tan^{-1}(\frac{\gamma(k)}{\lambda}) + \sigma_{2,1}(k)$$

So large $\ell$ behavior of $\sigma_{2}$ is:

$$\sigma_{2}(k) \rightarrow \frac{\gamma(k)}{\lambda} + \sigma_{2,1}(k)$$

Solution to iterative asymptotic eqn:

$$\sigma_{2}(k) \rightarrow \sigma_{2,\infty}(k) + \sum_{n=0}^{\lambda} \frac{\gamma(k)}{n} \int_{1}^{\lambda} dn \frac{\gamma(k)}{n}$$

$$\sigma_{2}(k) = \sigma_{2,\infty}(k) + \gamma(k) \ln(k) \quad \text{as} \quad k \rightarrow \infty$$

$$\therefore f_{2,1}^{\text{coul}}(k) \rightarrow \frac{1}{2i\kappa} \left( e^{2i(\sigma_{0} + \gamma \ln k)} - 1 \right)$$

highly oscillatory with ampl bounded

$$|f_{2}(k)| \leq \frac{1}{k} \quad \text{indep of} \quad k.$$
2) Divergence of projection integral:

\[ f_{2}^{\text{proj}}(k) = \frac{1}{2} \int_{-\pi}^{\pi} d(\cos \theta) f_{2}^{\text{proj}}(k', k) P_{2}(\cos \theta) \]

Write scattering ampl. in terms of \( \cos \theta \):

\[ f_{2}^{\text{proj}}(k', k) = -\gamma \frac{e^{2i\phi}\gamma}{2k \sin^{2} \theta/2} \]

\[ = -\gamma e^{2i\phi}\frac{\sin ^{2}(\frac{\theta}{2})^{-i\gamma}}{2k \sin^{2} (\frac{\theta}{2})} \]

\[ = -\gamma e^{2i\phi} \frac{e^{-i\gamma} \ln 2}{k} (1 - \cos \theta)^{-1-i\gamma} \]

\[ \text{det} : = P(\theta) \]

So,

\[ f_{2}^{\text{proj}}(k) \frac{1}{2} P(k) \int_{-\pi}^{\pi} d\theta (1 - \cos \theta)^{-1-i\gamma} P_{2}(\theta) \]

Integrab has a singularity at \( \theta = \pi \) (upper limit of integral)

\( P_{2}(\theta) \) is regular for all \( \theta \), so analyze for \( \theta = \pi \).

\[ \lim_{\epsilon \to 0} \int_{-\pi}^{1-\epsilon} d\theta (1 - \epsilon)^{-1-i\gamma} = \frac{(1-\epsilon)^{-i\gamma}}{i\gamma} \mid_{-1}^{1-\epsilon} \]

\[ = \frac{2^{-i\gamma}}{i\gamma} + \lim_{\epsilon \to 0} \frac{1 - (1-\epsilon)^{-i\gamma}}{i\gamma} \]

\[ = \frac{2^{-i\gamma}}{i\gamma} + \frac{1}{i\gamma} \lim_{\epsilon \to 0} e^{-i\gamma} \]

\[ = \frac{2^{-i\gamma}}{i\gamma} + \frac{1}{i\gamma} \lim_{\epsilon \to 0} \left[ \cos (\gamma \ln \epsilon) - i \sin (\gamma \ln \epsilon) \right] \]

No limit!
Conditions under which partial wave series converges —

as a distribution.

Consider the equality we wish to view as a distribution:

\[ \sum_{j=0}^{\infty} (2j+1) f_j^p(k') P_j(\cos \theta) = f(k' \mapsto k) \]

Multiply both sides by smooth test function \( \varphi(\cos \theta) \):

\[ \sum_{j=0}^{\infty} (2j+1) f_j^p(k') P_j(\cos \theta) \varphi(\cos \theta) = f(k' \mapsto k) \varphi(\cos \theta) \]

Then integrate over \( \cos \theta \) [keep \( 2j+1 \) under integral]

\[ \sum_{j=0}^{\infty} \left( (2j+1) f_j^p(k') \int_{-1}^{1} P_j(\cos \theta) \varphi(\cos \theta) d \cos \theta \right) = \int_{-1}^{1} f(k' \mapsto k) \varphi(\cos \theta) d \cos \theta \]

Define \( g_j = \int_{-1}^{1} d z \ (2j+1) \varphi(z) P_j(z) \).

Note: \( \sum_{j=0}^{\infty} g_j = \int_{-1}^{1} \varphi(z) \sum_{j=0}^{\infty} (2j+1) P_j(z) = 2 \delta(1-z) \).

LHS

RHS

\[ \sum_{j=0}^{\infty} g_j f_j^p(k') = \int_{-1}^{1} f(k' \mapsto k) \varphi(\cos \theta) d \cos \theta \]

Expect this to hold as long as

\( \varphi(\cos \theta) \) is twice differentiable on interval \( -1 \leq \cos \theta \leq 1 \),

\[ \varphi'(1) = 0 \]

vanishes at forward angle.

Taylor proves LHS is convergent, \( \varphi(z) \) the value it converges to is RHS,

Take Rhs, and write as limit:

\[ \text{RHS} = \lim_{\epsilon \to 0} \int_{-1}^{1-i\epsilon} f(k' \mapsto k) \varphi(\cos \theta) d \cos \theta \]
Then expand \( Y(\cos \theta) \) as a series in Legendre \( P \),

(also because \( Y \) is smooth on interval \(-1, \cos \theta \leq 1\))

\[
Y(\cos \theta) = \frac{1}{2} \sum_{j=0}^{\infty} P_j(\cos \theta) P_j(\cos \theta)
\]

So that

\[
\text{RHS} = \lim_{\epsilon \to 0} \int_{-1}^{1-\epsilon} f(\cos \theta) d(\cos \theta) = \frac{1}{2} \sum_{j=0}^{\infty} a_j \int_{-1}^{1-\epsilon} P_j(\cos \theta) d(\cos \theta)
\]

Interchange sum and integral

(allowed because sum and integrals are uniformly convergent)

\[
= \lim_{\epsilon \to 0} \sum_{j=0}^{\infty} a_j \int_{-1}^{1-\epsilon} P_j(\cos \theta) d(\cos \theta)
\]

To match RHS, this reads to be \( f_1(\cos \theta) \) doesn't work due to \( \lim_{\epsilon \to 0} \) both

But we know this cannot be, for the \( \epsilon \to 0 \) limit does not exist

for partial sum projection of \( f \) because of the singularity of \( f(\cos \theta) \) at \( \cos \theta = 1 \).

But difference between two adjacent projections does converge:

\[
\frac{1}{2} \int_{-1}^{1-\epsilon} f(\cos \theta) \left( P_j(\cos \theta) - P_{j-2}(\cos \theta) \right) d(\cos \theta) = f_{j-2}^{\text{conv}}(k) - f_{j-2}^{\text{conv}}(k)
\]

\[
\uparrow
\]

\[
\text{difference cancels at } \cos \theta = 1
\]

the integral can be evaluated with the help of Integral tables
to give:

\[
f_j^{\text{conv}}(k) - f_{j-2}^{\text{conv}}(k)
\]

Then we have the following four of relationship:

\[
a_j - a_{j-2} \to f_j - f_{j-2}
\]

\[
a_{j-2} - a_{j-4} \to f_{j-2} - f_{j-4}
\]

\[
\vdots
\]

\[
a_2 - a_0 \to f_2 - f_0 \Rightarrow a_j - (a_0 - f_0) \to f_j
\]
\[
a_0 = \frac{1}{2} \int_{-1}^{1} f e^{j \theta} \, d(\cos \theta) \\
f_u = f_{\theta=0}
\]

So go back to RHS, and must \((a_0 - f_0) \sum \Phi_k = 0\)

\[
RHS = \lim_{\epsilon \to 0} \sum_{j=0}^{\infty} \Phi_k \left[ \frac{1}{2} \int_{-1}^{1-\epsilon} f e^{j \theta} P_2(\cos \theta) \, d(\cos \theta)
- \left( \frac{1}{2} \int_{-1}^{1-\epsilon} f' e^{j \theta} \, d(\cos \theta) - f_{\theta=0}(k) \right) \right]
\]

Then interchange limit and sum

\[
= \sum_{j=0}^{\infty} \Phi_k \lim_{\epsilon \to 0} \left[ \frac{1}{2} \int_{-1}^{1-\epsilon} f e^{j \theta} P_2(\cos \theta) \, d(\cos \theta)
- \left( \frac{1}{2} \int_{-1}^{1-\epsilon} f' e^{j \theta} \, d(\cos \theta) - f_{\theta=0}(k) \right) \right]
= \sum_{j=0}^{\infty} \Phi_k f_k(k) = LHS
\]

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**Point:** The projection of Coulomb amplitude onto a single partial wave is not possible — singular.

But it is possible to project amplitude onto a difference of adjacent partial waves, and therefore onto a difference of any pair of partial waves.