Analyticity of $S_k(k)$: (for typical potentials, not exponentially vanishing at $r \to \infty$)

$\hat{\phi}_k(k)$ analytic in upper-half plane $\implies$ no region of analyticity

$\hat{\phi}_k(-k)$ analytic in lower-half plane $\implies$ for $S(k) = \frac{\hat{\phi}_k(-k)}{\hat{\phi}_k(k)}$?

Can do better: can establish analyticity in the lower half plane:

Since $\hat{\phi}_k(k,r)$ (and $\hat{\phi}_k^+(kr)$ & $U(r)$) analytic for $r > 0$,
can rotate contour integral (arc contribution vanishes)

$\hat{\phi}_k(k) = 1 + \frac{1}{k} \int_0^\infty \, dr \, \hat{h}_k^+(kr) \, U(r) \, \phi_k(k,r)$

$= 1 + \frac{1}{k} \int_0^{\infty} \, dr \, \hat{h}_k^+(kr) \, U(r) \, \phi_k(k,r)$

Choose integration variables: $r \to r e^{-i\theta}$

$= 1 + \frac{e^{-i\theta}}{k} \int_0^{\infty} \, dr \, \hat{h}_k^+(k r e^{-i\theta}) \, U(r e^{-i\theta}) \, \phi_k(k, r e^{-i\theta})$

Then, bounds become:

$|\hat{\phi}_k(k, r) - 1| \leq \text{const.} \left| \frac{1}{k} \int_0^{\infty} \, dr \, \frac{|k r e^{-i\theta}|}{1 + |k r e^{-i\theta}|} \right| U(r e^{-i\theta}) e^{-i\theta} = |\text{Im}(k r e^{-i\theta})| + |\text{Im}(k r e^{-i\theta})|

\text{Im}(k r e^{-i\theta}) = -r \text{Im}(k e^{-i\theta})$

So, integral converges if $\text{Im}(k e^{-i\theta}) > 0$

$\implies \hat{\phi}_k(k)$ analytic for $\text{Im}(k e^{-i\theta}) > 0$.

And since $\hat{\phi}_k^+, \hat{\phi}_k^+ \& U(r)$ analytic for $\Theta$
satisfying $-\pi/2 < \Theta < \pi/2$,
union of regions covered by $S \implies$

$\hat{\phi}_k(k)$ analytic for all $k$, except for a cut down the negative imaginary axis.
Analyticity of Jost function for various potentials:

\[ \frac{f_+(-k)}{f_+(k)} \]

For \( U(r \to \infty) \sim \frac{1}{r^{2-\delta}} \)

For \( U(r \to \infty) \sim e^{-\mu r} \)

For \( U(r \to \infty) = 0 \) (truncated potentials)

So \( S \)-matrix elements, \( S_k(k) = \frac{f_+(k)}{f_+(k)} \) are meromorphic - analytic except for the existence of isolated poles, \( k \) due to zeros of \( f_+(k) \).

\[ \frac{S_+(k)}{S_+(k)} \]

For \( U(r \to \infty) \sim \frac{1}{r^{2-\delta}} \)

For \( U(r \to \infty) \sim e^{-\mu r} \)

For \( U(r \to \infty) = 0 \)