Invariant amplitude approach

Coefficient functions transform covariantly under rotations.

→ Allows easy implementation of discrete symmetries.

For (spin 0) - (spin 1/2) scattering:

Write amplitude matrix in terms of \( \mathbb{1}, \sigma, \kappa, \kappa' \) and \( \hat{n} = \frac{\kappa \times \kappa'}{|\kappa \times \kappa'|} \) and \( E = |\mathcal{E}|^2 \).

\[
\hat{f}_{3,3}^\pm (E, \theta, \phi) = \begin{pmatrix}
a(E, \theta) & e^{-i \phi} b(E, \theta) \\
c^* b(E, \theta) & d(E, \theta)
\end{pmatrix} = \begin{pmatrix}
a & ib(-\cos \phi - \sin \phi) \\
0 & d
\end{pmatrix}
\]

Spin quantized in direction of \( \kappa' \) (formerly the z-axis).

One possibility:

\[
F = \mathbb{1} A(E, \frac{\mathcal{E}}{2}) + \tilde{\kappa} \sigma B(E, \frac{\mathcal{E}}{2}) + \kappa' \sigma C(E, \frac{\mathcal{E}}{2}) + \hat{n} \cdot \sigma D(E, \frac{\mathcal{E}}{2})
\]

a more prudent choice

(for covariance under time reversal)

\[
F = \mathbb{1} A(E, \frac{\mathcal{E}}{2}) + (\kappa + \kappa') \cdot \sigma B(E, \frac{\mathcal{E}}{2}) + (\kappa' - \kappa) \cdot \sigma C(E, \frac{\mathcal{E}}{2}) + \hat{n} \cdot \sigma D(E, \frac{\mathcal{E}}{2})
\]

Coefficient functions \( A, B, C, D \) are the invariant amplitudes (under rotation).

\( E \sim \mathcal{E}^2 \) and \( \frac{\mathcal{E}}{2} \sim \kappa' \kappa' \) combine \( \theta \) dep.

To match \( F \) against \( \hat{f}_{3,3}^\pm \), orient \( \kappa \) along z-axis

\[
\hat{\kappa} = (0, 0, 1)
\]

\[
\kappa' = \begin{pmatrix}
\sin \theta & 0 & \cos \theta
\end{pmatrix} \quad \text{if } \phi = 0
\]

\[
= \begin{pmatrix}
\sin \theta & 0 & \cos \theta
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \phi \sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta
\end{pmatrix} \quad \text{if } \phi \neq 0.
\]

\[
\hat{\kappa} \times \hat{\kappa}' = \begin{pmatrix}
-\sin \phi \sin \theta & \cos \phi \sin \theta & 0
\end{pmatrix} \Rightarrow |\hat{\kappa} \times \hat{\kappa}'| = \sin \theta
\]

\[
\hat{n} = (-\sin \phi, \cos \phi, 0)
\]
\[ F = A + \hat{\mathbf{k}} \cdot \mathbf{\sigma}(B + C) + \hat{\mathbf{k}} \cdot \mathbf{\sigma}(B - C) + i\hat{\mathbf{n}} \cdot \mathbf{\sigma} D \]

\[
\hat{\mathbf{k}} \cdot \mathbf{\sigma} = \begin{pmatrix} \cos \theta & (\cos \phi - i \sin \phi) \sin \theta \\ (\cos \phi + i \sin \phi) \sin \theta & -\cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix}
\]

\[
\hat{\mathbf{n}} \cdot \mathbf{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
i\hat{\mathbf{n}} \cdot \mathbf{\sigma} = i \begin{pmatrix} 0 & -\sin \phi - i \cos \phi \\ -\sin \phi + i \cos \phi & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix}
\]

So,

\[
F = \begin{pmatrix} A + (B - C) + (B + C) \cos \theta & e^{-i\phi}((B + C) \sin \theta + D) \\ e^{i\phi}((B + C) \sin \theta - D) & A - (B - C) - (B + C) \cos \theta \end{pmatrix}
\]

\[ a(E, \theta) = A + (B - C) + (B + C) \cos \theta \]
\[ b(E, \theta) = (B + C) \sin \theta + D \]
\[ c(E, \theta) = (B + C) \sin \theta - D \]
\[ d(E, \theta) = A - (B - C) - (B + C) \cos \theta \]

Inverse relations:

\[ A(k^2, q^2) = \frac{1}{2} (a + d) \]
\[ B(k^2, q^2) = \frac{1}{4} (a - d + (b + c) \tan \frac{\theta}{2}) \]
\[ C(k^2, q^2) = \frac{1}{4} (-a + d + (b + c) \tan \frac{\theta}{2}) \]
\[ D(k^2, q^2) = \frac{1}{2} (b - c) \]
Invariance under discrete symmetries

\[ F = 1 A + (\vec{r}+\vec{r}') \cdot \vec{\sigma} \cdot B + (\vec{r}-\vec{r}') \cdot \vec{\sigma} \cdot C + i \hat{n} \cdot \vec{\sigma} \cdot D \]

(1) Parity invariance:

\[
\begin{aligned}
\vec{r} &\quad \xrightarrow{P} \quad -\vec{r}' \\
\vec{r}' &\quad \xrightarrow{P} \quad -\vec{r} \\
\hat{n} &\quad \xrightarrow{P} \quad \hat{n}
\end{aligned}
\]

\[ \Rightarrow B = C = 0 \]

\[ F = 1 A + i \hat{n} \cdot \vec{\sigma} \cdot D \]

(Parity invariance)

(2) Time reversal invariance:

\[
\begin{aligned}
\vec{r} &\quad \xrightarrow{T} \quad -\vec{r}' \\
\vec{r}' &\quad \xrightarrow{T} \quad -\vec{r} \\
\vec{\sigma} &\quad \xrightarrow{T} \quad -\vec{\sigma} \\
\hat{n} &\quad \xrightarrow{T} \quad -\hat{n}
\end{aligned}
\]

\[ \Rightarrow F \rightarrow \quad 1 A + (\vec{r}'+\vec{r}) \cdot \vec{\sigma} \cdot B + (\vec{r}'-\vec{r}) \cdot \vec{\sigma} \cdot C + \hat{n} \cdot \vec{\sigma} \cdot D \]

\[ \text{sign change } \Rightarrow C = 0 \]

\[ F = 1 A + (\vec{r}'+\vec{r}) \cdot \vec{\sigma} \cdot B + i \hat{n} \cdot \vec{\sigma} \cdot D \]

(Time-reversal invariance)

For \((\text{spin 0})-(\text{spin } 1/2)\) scattering with parity invariance, time reversal invariance provides no new constraints.