Spherical harmonics - summary

Equation 1:

\[
\frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \left( \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right) Y_{\ell m}^m(\theta, \phi) = -\ell(\ell+1) Y_{\ell m}^m(\theta, \phi)
\]

\[
\frac{\partial}{\partial \phi} Y_{\ell m}^m(\theta, \phi) = im \ Y_{\ell m}^{m}(\theta, \phi)
\]

Separation ansatz: \( Y_{\ell m}^m(\theta, \phi) = f_{\ell m}^m(\theta) \ g_{\ell m}^m(\phi) \)

\[
\Rightarrow \ f_{\ell m}^m(\theta) \frac{\partial}{\partial \phi} \ g_{\ell m}^m(\phi) = im \ f_{\ell m}^m(\theta) \ g_{\ell m}^m(\phi)
\]

\[
\Rightarrow \ g_{\ell m}^m(\phi) = e^{im\phi} \ \text{indep. of} \ \ell
\]

Boundary condition: \( g_{\ell m}^m(\phi) = g_{\ell m}^m(\phi + 2\pi) \Rightarrow m \in \mathbb{Z} \) (cyclic)

Equation 2:

\[
\left[ \frac{-m^2}{\sin^2 \theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right] f_{\ell m}^m(\theta) \ g_{\ell m}^m(\phi) = -\ell(\ell+1) f_{\ell m}^m(\theta) \ g_{\ell m}^m(\phi)
\]

ODE

\[
\left[ \frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d}{d \theta} \right) - \frac{m^2}{\sin^2 \theta} + \ell(\ell+1) \right] f_{\ell m}^m(\theta) = 0
\]

Path: \( \cos \theta = z \Rightarrow \sin \theta = \sqrt{1-z^2} \) and \( \frac{d}{d \theta} = \sqrt{1-z^2} \frac{d}{dz} \)

so that

\[
\left[ \frac{1}{\sqrt{1-z^2}} \frac{d}{dz} \left( \sqrt{1-z^2} \frac{d}{dz} \right) \right] - \frac{m^2}{1-z^2} + \ell(\ell+1) \ f_{\ell m}^m(z) = 0
\]

\[
(1-z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{m^2}{1-z^2} + \ell(\ell+1) \ f_{\ell m}^m(z) = 0.
\]

(well-known)

Solution: \( f_{\ell m}^m(z) = P_{\ell m}^m(z) \) (associated Legendre polynomials).
Summary

\[ Y_{l}^{m}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{l}^{m}(\cos \theta) e^{im\phi} \]

In Mathematics

Spherical Harmonic \( Y_{l,m}^{m}(\theta, \phi) \)

\[ P_{l}^{m}(z) = (-1)^{m} (1-z^{2})^{m/2} \frac{d^{m}}{dz^{m}} P_{l}(z) \]

Legendre \( P_{l,m}(\theta, \phi) \)

\[ P_{l}^{m}(z) = (-1)^{m} \frac{(l+ml)!}{(l-ml)!} P_{l}^{ml}(z) \]

Legendre \( P_{l,m}(\theta, \phi) \)

Condon-Shortley phase

Both cases:

\[ Y_{l}^{m}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \times \left\{ \begin{array}{ll} P_{l}^{m}(\cos \theta) e^{im\phi} & \text{if } l \geq m \geq 0 \\ P_{l}^{ml}(\cos \theta) e^{im\phi} & \text{if } l > m \geq 0 \end{array} \right. \]

Orthogonality:

\[ \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta \ Y_{l}^{m}(\theta, \phi) Y_{l}^{m'}(\theta, \phi) = \delta_{mm'} \delta_{ll'} \]

Completeness:

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{l}^{m}(\theta, \phi) Y_{l}^{m'}(\theta, \phi') = \frac{1}{\sin \theta} \delta(\theta - \theta') \delta(\phi - \phi') \]

Addition theorem:

\[ \sum_{m=-l}^{l} Y_{l}^{m}(\theta', \phi') Y_{l}^{m}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} Y_{l}^{0}(\theta, 0) = \frac{2l+1}{4\pi} P_{l}(\cos \theta)} \]

Direction angle for vector 1 and vector 2

\[ \text{between two vectors} \]

\[ \text{can be any value} \]

\[ \text{(doesn't matter)} \]

Useful reminder: angles inside \( Y_{l}^{m}(\theta, \phi) \) are measured against the axis of angular momentum quantization described by the \((l,m)\) numbers.