Scalar Ca function in 4-2ε dimensions:

\[
\begin{align*}
\mu^2 e \int \frac{d^4k}{(2\pi)^4} & \frac{1}{e^{2m_0^2/\hbar} + 2\epsilon} \frac{1}{e^{(k+p_2)^2-m_1^2/\hbar}} \frac{1}{e^{(k+p_3)^2-m_2^2/\hbar}} \\
& \text{Independent invariants:} \\
& \frac{p_2^2}{q^2}, \quad \frac{p_2^2 - p_1^2}{q^2}, \quad p_2^2 \\
& \text{Integral to do:} \\
& \mu^2 e \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{2m_0^2/\hbar} + i\epsilon} \frac{1}{e^{(k+p_2)^2-m_1^2/\hbar} + i\epsilon} \frac{1}{e^{(k+p_3)^2-m_2^2/\hbar} + i\epsilon} \\
& \text{Combine denominators:} \\
& = \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(1-x-y-z) \mu^2 e \int \frac{d^4k}{(2\pi)^4} \frac{2}{\cdots} \\
& \left[ \cdots \right] = x(k^2 - m_0^2) + y((k+p_2)^2 - m_1^2) + z((k+p_3)^2 - m_2^2) + i\epsilon \\
& = (x+y+z) k^2 + 2k \cdot (y p_2 + z p_3) + y p_1^2 + z p_2^2 - x m_0^2 - y m_1^2 - z m_2^2 + i\epsilon \\
& \text{Denominator:} \\
& = k^2 + 2k \cdot Q + A^2 + i\epsilon \\
& = k^2 + 2k \cdot Q + A^2 + i\epsilon \\
& \mu^2 e \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{2m_0^2/\hbar} + i\epsilon} \frac{1}{e^{(k+p_2)^2-m_1^2/\hbar} + i\epsilon} \frac{1}{e^{(k+p_3)^2-m_2^2/\hbar} + i\epsilon} \\
& \text{Integrate over } k.
\[
= \int_0^1 dx \int_0^1 dy \int_0^1 dz \: \frac{\mu^2 e^{-i z} \gamma(3-d/2)}{(y^2 + p^2 + z^2 \cdot p^2)} \\
\quad \times y^2 \cdot p^2 + 2yz \cdot p \cdot p_z + z^2 \cdot p_z^2 + (1-y-z) m_0^2 - y(p_1^2 - m_1^2) - z(p_2^2 - m_2^2) - i \epsilon \\
= \frac{i e^{-\gamma(1+\epsilon)} \gamma(1+\epsilon)}{(4\pi)^{3/2}} \int_0^1 dy \int_0^1 dz \left( \frac{1}{\Delta(y,z)} \right)^{1+\epsilon} \]

\text{Denominator function:}
\[
\Delta(y,z) = y^2 \cdot p_1^2 + 2yz \cdot p_1 \cdot p_2 + z^2 \cdot p_2^2 + (1-y-z) m_0^2 - y(p_1^2 - m_1^2) - z(p_2^2 - m_2^2) - i \epsilon \\
= p_1^2 \cdot y^2 + p_2^2 \cdot z^2 + 2p_1 \cdot p_2 \cdot yz + (-p_1^2 + m_1^2 - m_0^2) y \\
\quad + (-p_2^2 + m_2^2 - m_0^2) z + m_0^2 - i \epsilon \\
\quad \text{while} \quad 2p_1 \cdot p_2 = -(p_1 - p_2)^2 + p_1^2 + p_2^2 \\
\Delta(y,z) = p_1^2 \cdot y^2 + p_2^2 \cdot z^2 + (-p_1^2 + p_1^2 + p_2^2) yz + (-p_1^2 + m_1^2 - m_0^2) y \\
\quad + (-p_2^2 + m_2^2 - m_0^2) z + m_0^2 - i \epsilon \\
\]

More convenient to focus on combinations \((1-y)\) instead of \(y\).

\text{cho. integration variables:}
\[
\begin{align*}
y &= 1 - y' & dy &= - dy' \\
y' &= 1 - y & \text{range:} \quad y: 0 \rightarrow 1 \\
y': 1 \rightarrow 0 & \text{flip/cancel.}
\end{align*}
\]
\[ C_0 = -\mu_2^2 e^{\eta e} \Gamma(1+\varepsilon) \int_0^1 dy' \int_0^{y'} dz \left( \frac{1}{\Delta(1-y', z)} \right)^{1+\varepsilon} \]

where

\[ \Delta(1-y', z) = p_1^2 y'^2 + p_2^2 z^2 + (q^2 - p_1^2 - p_2^2) y'z \]

\[ + (-p_1^2 + m_0^2 - m_1^2) y' + (p_1^2 - q^2 - m_0^2 + m_2^2) z + m_2^2 - i \varepsilon \]

- drop primes on \( y \).

So the integral we need to do has the form:

\[ C_0 = -\mu_2^2 e^{\eta e} \Gamma(1+\varepsilon) \int_0^1 dy \int_0^y dz \left( \frac{1}{ay^2 + b z^2 + cyz + dy + ez + f} \right)^{1+\varepsilon} \]

where:

- \( a = p_1^2 \)
- \( b = p_2^2 \)
- \( c = q^2 - p_1^2 - p_2^2 = -2p_1p_2 \)
- \( q^2 = a + b + c \)
- \( d = -p_1^2 + m_0^2 - m_1^2 \)
- \( m_0^2 = a + d + f + i \varepsilon \)
- \( e = p_1^2 - q^2 - m_0^2 + m_1^2 \)
- \( m_1^2 = f + i \varepsilon \)
- \( f = m_0^2 - i \varepsilon \)
- \( m_2^2 = a + b + c + d + e + f + i \varepsilon \)

If all \( p_1^2, p_2^2, q^2, m_0, m_1, m_2 \) are non-zero, integral is \( \mathbb{R} \) finite. Can take \( \varepsilon \to 0 \) limit:

\[ C_0(p_1^2, q^2, p_2^2, m_0, m_1, m_2) = -\int_0^1 dy \int_0^y dz \frac{1}{ay^2 + b z^2 + cyz + dy + ez + f}. \]

Put minus sign on other side, and proceed to integrate.
Hiren H. Patel

\[ C_0 = \int_0^1 dy \int_{-\alpha y}^{(1-\alpha)y} \frac{1}{(b \alpha^2 + c \alpha + d) y^2 + b z^2 + (c+2 \alpha b) y z + (d+e \alpha) y + e \alpha + f} dz \]

where \( \alpha = \frac{-c \pm \sqrt{c^2 - 4ab}}{2b} \)

Then, split integral into region above \( y \)-axis

and below \( y \)-axis.

Reparameterize so that integral over \( y \) is first.
\[-C_0 = \left( \int_{0}^{1-\alpha} dz \int_{\frac{y}{1-\alpha}}^{1} dy - \int_{0}^{-\alpha} dz \int_{\frac{y}{\alpha}}^{1} dy \right) \frac{1}{b z^2 + (c + 2 b \alpha) y z + (d + e \alpha) y + e z + f} \]

Organize denominator by powers of \( y \):

\[
\text{Denom} = \left[ \frac{y (c + 2 b \alpha) + d + e \alpha}{b z^2 + e z + f} \right] y + \frac{1}{b z^2 + e z + f}
\]

\[= F(z) y + G(z)\]

The \( y \) integral can now be done:

\[-C_0 = \int_{0}^{1-\alpha} dz \int_{\frac{y}{1-\alpha}}^{1} \frac{dy}{F(z) y + G(z)} - \int_{0}^{-\alpha} dz \int_{\frac{y}{\alpha}}^{1} \frac{dy}{F(z) y + G(z)}\]

\[= \int_{0}^{1-\alpha} dz \left[ \frac{\ln(F(z) y + G(z))}{F(z)} \right]_{\frac{y}{1-\alpha}}^{1} - \int_{0}^{-\alpha} dz \left[ \frac{\ln(F(z) y + G(z))}{F(z)} \right]_{\frac{y}{\alpha}}^{1}\]

Integration contours run parallel to cuts:

- will never cross cut \( \Rightarrow \) safe to split \( \ln \) arguments

\[-C_0 = \int_{0}^{1-\alpha} dz \frac{\ln(F(z) + G(z)) - \ln\left(\frac{z}{1-\alpha} F(z) + G(z)\right)}{F(z)}\]

\[= \int_{0}^{-\alpha} dz \frac{\ln(F(z) + G(z)) - \ln\left(\frac{z}{\alpha} F(z) - G(z)\right)}{F(z)}\]

\[= \int_{0}^{1-\alpha} dz \frac{\ln(F(z) + G(z)) - \ln\left(\frac{z}{1-\alpha} F(z) + G(z)\right)}{F(z)}\]

Combine first term of each integral - they have same integrand,

\[-C_0 = \int_{-\alpha}^{1-\alpha} dz \frac{\ln(F(z) + G(z))}{F(z)} - \int_{0}^{1-\alpha} dz \frac{\ln\left(\frac{z}{1-\alpha} F(z) + G(z)\right)}{F(z)} + \int_{0}^{-\alpha} dz \frac{\ln\left(\frac{z}{\alpha} F(z) + G(z)\right)}{F(z)}\]
\[ -C_0 = \int_{-\alpha}^{1-\alpha} \frac{\ln (F(z) + G(z))}{F(z)} \, dz - \int_{0}^{1-\alpha} \frac{\ln \left( \frac{z}{1-\alpha} F(z) + G(z) \right)}{F(z)} \, dz \\
+ \int_{0}^{-\alpha} \frac{\ln \left( \frac{-z}{\alpha} F(z) + G(z) \right)}{F(z)} \, dz \]

Notice: the denominators lead to a pole when \( F(z_0) = 0 \)

\[ F(z_0) = 0 \quad \text{when} \quad z_0 = \frac{d + e \alpha}{c + 2b \alpha} \]

But, even though each term might be singular as \( z \to z_0 \),
the arguments of the logs in all three terms are the same:

\[ \ln (F(z) + G(z)) \xrightarrow{z \to z_0} \ln G(z_0) \quad \text{because} \quad F(z_0) = 0. \]
\[ \ln \left( \frac{Z}{1-\alpha} F(z) + G(z) \right) \xrightarrow{z \to z_0} \ln G(z_0) \]
\[ \ln \left( -\frac{Z}{\alpha} F(z) + G(z) \right) \xrightarrow{z \to z_0} \ln G(z_0) \]

For a given \( z_0 \), only one or the other of the last two integrals will
hit the pole at \( Z = z_0 \) (this is because the ranges of these integrals are exclusive, except at \( z = 0 \)). And, the singularity will cancel against
the one from the first integral.

Each integral can be made manifestly regular at \( Z = z_0 \) by
subtracting \( \ln G(z_0) \) from the first term, and adding to the
last two terms,

\[ -C_0 = \int_{-\alpha}^{1-\alpha} \frac{\ln (F(z) + G(z)) - \ln G(z_0)}{F(z)} \, dz \]
\[ \int_{0}^{1-\alpha} \frac{\ln \left( \frac{Z}{1-\alpha} F(z) + G(z) \right) - \ln G(z_0)}{F(z)} \, dz + \]
\[ \int_{0}^{-\alpha} \frac{\ln \left( -\frac{Z}{\alpha} F(z) + G(z) \right) - \ln G(z_0)}{F(z)} \, dz \]
Stretch ranges to $0 \rightarrow 1$ by changing integration variables:

1st integral:  
$z' = z' - v$  
$z = (1 - v)(1 - z')$  
$z = -v z'$

$z' = z + v$  
$z' = 1 - \frac{z}{1 - v}$  
$z' = -v v$

range:  
$z: -v \rightarrow 1 - v$  
$z: 0 \rightarrow 1 - v$  
$z: 0 \rightarrow -v$

$z': 0 \rightarrow 1$  
$z': 1 \rightarrow 0$  
$z': 0 \rightarrow 1$

d$z = d z'$  
$d z = \int \frac{d z'}{F(z')}$  
$d z = -v d z'$

$- C_0 = \int_0^1 d z' \ln \left[ \frac{F(z') - z' G(z')}{F(z') - v G(v)} \right] \ln G(z_0)$

$- \int_0^1 (1 - v) d z' \ln \left[ \frac{((1 - z')(1 - z')) F((1 - v)(1 - z')) + G((1 - v)(1 - z'))}{F((1 - v)(1 - z'))} \right] \ln G(z_0)$

$+ \int_0^1 \frac{d z'}{F(-v z')} \ln \left[ \frac{(-v z') - z' G(-v z')}{F(-v z')} \right] - \ln G(z_0)$

Insert explicit expressions for $F$ and $G$

- Inside logs, $\alpha = \frac{c \pm \sqrt{c^2 - 4ab}}{2b}$ can be substituted to achieve drastic simplification. (Doesn't depend on $\pm$)

- The denominators, on the other hand, are complicated.

Don't insert explicit formula for $\alpha$ yet.

$- C_0 = \int_0^1 d z \ln \left( \frac{b z^2 + (c + e) z + a + d + f}{(c + 2b z)(z + d + c + e)} \right) - \ln G(z_0)$

$- \int_0^1 (1 - v) d z \ln \left[ \frac{(a + b + c) z^2 - (2a + 2b + 2c + d + e) z + (a + b + c + d + e + f)}{(1 - v)(c + 2b z)(1 - z) + d + e z} \right] - \ln G(z_0)$

$+ \int_0^1 \alpha d z \ln \left( \frac{a z^2 d z' + f}{a z (c + 2b z) + d + e \alpha} \right)$
Factor out $c+2b\alpha$ from denominator of each integral.

and multiply coeff. $1-\alpha$ and $\alpha$ into denominator in 2nd and 3rd integrals

$$-C_0 = \frac{1}{c+2b\alpha} \int_0^1 dz \frac{\ln \left( b z^2 + (c+e) z + a + d + f \right) - \ln G(z_0)}{(z + \frac{d+ae}{c+2b\alpha} - \alpha)}$$

$$+ \frac{1}{c+2b\alpha} \int_0^1 dz \frac{\ln \left( (a+b+c) z^2 - (2a+2b+2c+d+e) z + (a+b+c+d+e+f) \right) - \ln G(z_0)}{(z - 1 - \frac{d+ae}{c+2b\alpha})}$$

$$+ \frac{1}{c+2b\alpha} \int_0^1 dz \frac{\ln (az^2 + dz + f) - \ln G(z_0)}{(z - \frac{d+ae}{c+2b\alpha})}$$

Identity combinations $-\frac{d+ae}{c+2b\alpha} = z_0$

$$-C_0 = \frac{1}{c+2b\alpha} \int_0^1 dz \frac{\ln \left( b z^2 + (c+e) z + a + d + f \right) - \ln G(z_0)}{z - (z_0 + \alpha)}$$

$$+ \frac{1}{c+2b\alpha} \int_0^1 dz \frac{\ln \left( (a+b+c) z^2 - (2a+2b+2c+d+e) z + (a+b+c+d+e+f) \right) - \ln G(z_0)}{z - (1 - \frac{1}{1-\alpha} z_0)}$$

$$+ \frac{1}{c+2b\alpha} \int_0^1 dz \frac{\ln (az^2 + dz + f) - \ln G(z_0)}{z + z_0/\alpha}$$

Define: $z_2 = (z_0 + \alpha)$

$z_{12} = 1 - \frac{z_0}{1-\alpha}$ 

$z_1 = \frac{-z_0}{\alpha}$

New location of zero for which, when plugged into their respective $\ln(...)$, it equals $\ln G(z_0)$.

Now write argument of $\ln G(z_0)$ in terms of $z_1$, $z_2$, $z_{12}$.
Recall that in each integral, the subtraction of \( \ln G(z_o) \) was designed to make the singularity at \( z = z_o \) manifestly regular.

This came from the \( z \to z_o \) limit of new logs going into the second logs, giving a vanishing numerator.

Expect the same story here, but with \( z \to \{ z_1, z_2, z_{12} \} \).

Can check that

\[
G(z_o) = -\frac{ae^2 + b^2 - cde}{4ab - c^2} + f
\]

is equivalent to:

\[
= b z_2^2 + (c + e) z_2 + a + d + f = G(z_o) \text{ \ in terms of } z_2
\]

and

\[
= (a + b + c) z_{12}^2 - (2a + 2b + 2c + d + e) z_{12}
\]

\[
+ (a + b + c + d + e + f) = G(z_o) \text{ \ in terms of } z_{12}
\]

and

\[
= a z_1^2 + d z_1 + f = G(z_o) \text{ \ in terms of } z_1
\]

upon plugging in explicit formulas for \( z_1, z_{12}, z_2 \) and \( \alpha \).

Thus,

\[
-C_0 = \frac{1}{c + 2 ba} \int_0^1 dz \left\{ \frac{\ln \left( (b z_2^2 + (c + e) z + a + d + f) - \ln (z \to z_2) \right)}{z - z_2} \right.
\]

\[
+ \frac{\ln \left( (a + b + c) z_{12}^2 - (2a + 2b + 2c + d + e) z + (a + b + c + d + e + f) - \ln (z \to z_{12}) \right)}{z - z_{12}}
\]

\[
+ \frac{\ln (a z_1^2 + d z_1 + f) - \ln (z \to z_1)}{z - z_1} \right\}
\]
Put in momenta and masses for a, b, c, d, e, f:

$$b z^2 + (c+c) z + a + d + f$$

$$= p_z^2 z^2 - (p_z^2 - m_0^2 - m_z^2) z + m_0^2 - i \epsilon := P_z(z)$$

$$(a+b+c) z^2 - (2a+2b+2c+d+e) z + (a+b+c+d+e+f)$$

$$= \frac{q^2}{4} z^2 - (q^2 - m_1^2 + m_2^2) z + m_1^2 - i \epsilon := P_{12}(z)$$

$$a z^2 + d z + f = P_1^2 z^2 - \left( P_1^2 - m_0^2 + m_1^2 \right) z + m_1^2 - i \epsilon := P_1(z)$$

Then, 

$$-c_0 = \frac{1}{c+2b \alpha} \int_0^1 \frac{dz}{z} \left[ \ln P_0(z) \ln P_2(z) + \ln P_{12}(z) \ln P_{12}(z) + \ln P_1(z) \ln P_1(z) \right]$$

Subscripts explained: matched with pinch in corresponding channel.

$$c + 2b \alpha = c + 2b \left[ \frac{-c \pm \sqrt{c^2 - 4ab}}{2b} \right]$$

$$\alpha = \frac{1}{2b} \left[ -c \pm \sqrt{c^2 - 4ab} \right]$$

$$= \pm \sqrt{c^2 - 4ab}$$

$$= \pm \sqrt{(q^2 - p_1^2 - p_2^2) - 4 p_1^2 p_2^2}$$

$$= \pm \lambda \sqrt{p_1^2, p_2^2, q^2}$$

Note $z_1, z_2, z_{12}$ are complicated functions of momenta and masses.

$$z_1 = -\frac{z_0}{\alpha}$$

$$z_0 = -\frac{d + \epsilon}{c + 2b \alpha}$$

$$z_2 = z_0 + \alpha$$

$$z_{12} = 1 - \frac{z_0}{1 - \alpha}$$

But note: $P_1(z_1) = P_2(z_2) = P_{12}(z_{12}) = \frac{ae^2 + bd^2 - cde}{4ab - c^2} + f$
Reduction to Spence functions; for real masses and momenta in physical region $\lambda(p^2, p_i^2, q^2) > 0$.

- for case of no IR divergence.

Consider one integral of a given channel, $\mathcal{J} = \{1, 2, 12\}$

$$I_i = \int_0^1 dz \frac{\ln P(z) - \ln P(z_i)}{z - z_i}$$

where $z_i = \{z_1, z_2, z_{12}\} \in \mathbb{R}$

and $P(z) = a^2 z^2 + b z + c$ is a 2nd order polynomial.

Start by factorizing: $P_i(z) = \tilde{a} (z - z_{i+}) (z - z_{i-})$

where $z_{\pm} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ are the two roots of $P_i(z)$.

Recall result from $B_0$ function: We know that $z_{\pm}$ have opposite sign imaginary parts. => Split logarithms

$$= \int_0^1 dz \frac{1}{z - z_i} \left[ \ln \tilde{a} (z - z_{i+}) (z - z_{i-}) - \ln \tilde{a} (z_{i+} - z_{i+}) (z_{i-} - z_{i-}) \right]$$

Define integral as $R_i$, and integrate.

Notation: $z_{\pm} \equiv z_{i \pm}$ understood.
\[ R_{\pm} = \int_{0}^{1} \frac{\ln(z - z_{\pm}) - \ln(z_{i} - z_{\pm})}{z - z_{i}} \, dz \quad \text{n.b.: no pole at } z = z_{i} \]

\text{Note: branch point at } z = z_{\pm} \text{ may or may not fall inside range of integration.}

\text{sketches for } z_{\pm} \in \mathbb{R} \text{ (though could be } \in \mathbb{C})

In either case, shift integration variable to put branch point at } z = 0.

\[ z = z' + z_{\pm} \]

\text{range } z': \quad 0 \to 1 \quad \quad z': \quad -z_{\pm} \to 1 - z_{\pm}

\[ R_{\pm} = \int_{-z_{\pm}}^{1-z_{\pm}} \frac{\ln(z') - \ln(z_{i} - z_{\pm})}{z' + z_{\pm} - z_{i}} \, dz' \]

\text{split range into two parts}

\[ = \left[ \int_{0}^{1-z_{\pm}} - \int_{0}^{-z_{\pm}} \right] \frac{\ln(z') - \ln(z_{i} - z_{\pm})}{z' + z_{\pm} - z_{i}} \, dz' \]

\[ = \left[ \int_{0}^{1-z_{\pm}} - \int_{0}^{-z_{\pm}} \right] \frac{\ln(z') - \ln(z_{i} - z_{\pm})}{z' + z_{\pm} - z_{i}} \, dz' \]

\text{A} \quad \text{B} \quad \text{C}
Straighten out integration so that range is from $0 \to 1$

$1^{st}$ integral:

$Z' = (1 - Z_\pm) Z''$

$dz' = (1 - Z_\pm) dz''$

range $Z': 0 \to 1 - Z_\pm$

$Z' = -Z_\pm Z''$

$dz' = -Z_\pm dz''$

range $Z': 0 \to -Z_\pm$

$Z'': 0 \to 1$

$$R_\pm = \int_0^1 dz \frac{1}{1 - Z_\pm} \ln \left( \frac{(1 - Z_\pm) Z}{1 - Z_\pm} \right) - \ln \left( \frac{Z_\pm - Z}{Z_\pm + Z - Z_i} \right)$$

$$- \int_0^1 dz \frac{1}{Z_\pm Z + Z - Z_i} \left[ \ln \left( \frac{-Z_\pm Z}{Z_\pm + Z - Z_i} \right) - \ln \left( \frac{Z_\pm - Z}{Z_\pm + Z - Z_i} \right) \right]$$

Then:

$$\frac{1}{Z + a} = \frac{d}{dz} \ln (Z + a) = \frac{d}{dz} \ln \left( \frac{Z}{a} + 1 \right)$$

$$R_\pm = \int_0^1 dz \frac{d}{dz} \ln \left( 1 + \frac{1 - Z_\pm}{Z_\pm - Z_i} \right) \left[ \ln \left( 1 - Z_\pm Z \right) - \ln \left( Z_\pm - Z \right) \right]$$

$$- \int_0^1 dz \frac{d}{dz} \left[ \ln \left( 1 - \frac{Z_\pm}{Z_\pm - Z_i} \right) \right] \left[ \ln \left( -Z_\pm Z \right) - \ln \left( Z_\pm - Z \right) \right]$$

Integrate by parts.
\[ R_\pm = -\int_0^1 dz \ln \left( 1 + \frac{1 - z_i}{z_i - z_\pm} \right) z \left. \right|_{z_i}^{1} + \ln \left( 1 + \frac{1 - z_i}{z_i - z_\pm} \right) \ln \left( 1 - z_i \right) - \ln \left( z_i - z_\pm \right) \]

\[ + \int_0^1 dz \ln \left( 1 - \frac{z_i}{z_i - z_\pm} \right) z \left. \right|_{z_i}^{1} - \ln \left( 1 - \frac{z_i}{z_i - z_\pm} \right) \ln \left( z_i - z_\pm \right) \]

\[ = L_{i_2} \left( \frac{1 + z_i}{z_i - z_\pm} \right) + \ln \left( \frac{1 - z_i}{z_i - z_\pm} \right) \ln \left( 1 - z_i \right) - \ln \left( z_i - z_\pm \right) \]

\[ - L_{i_2} \left( \frac{z_i}{z_i - z_\pm} \right) - \ln \left( \frac{z_i}{z_i - z_\pm} \right) \ln \left( z_i - z_\pm \right) \]

Combine logarithm: \( \lambda \left( p_1, p_2, f_1, f_2 \right) > 0 \Rightarrow z_\pm \in \mathbb{R} \).

\[ \ln \left( 1 - z_\pm \right) - \ln \left( z_i - z_\pm \right) = \ln \left( \frac{1 - z_i}{z_i - z_\pm} \right) + \eta \left( 1 - z_\pm, \frac{1}{z_i - z_\pm} \right) \]

\[ \ln \left( z_i - z_\pm \right) - \ln \left( z_i - z_\pm \right) = \ln \left( \frac{z_i}{z_i - z_\pm} \right) + \eta \left( z_\pm, \frac{1}{z_i - z_\pm} \right) \]

\[ R_\pm = L_{i_2} \left( \frac{z_i}{z_i - z_\pm} \right) + \ln \left( \frac{1 - z_i}{z_i - z_\pm} \right) \ln \left( \frac{z_i - 1}{z_i - z_\pm} \right) + \ln \left( \frac{1 - z_i}{z_i - z_\pm} \right) \eta \left( 1 - z_\pm, \frac{1}{z_i - z_\pm} \right) \]

\[ - L_{i_2} \left( \frac{z_i}{z_i - z_\pm} \right) - \ln \left( \frac{z_i}{z_i - z_\pm} \right) \ln \left( \frac{z_i}{z_i - z_\pm} \right) - \ln \left( \frac{z_i}{z_i - z_\pm} \right) \eta \left( - z_\pm, \frac{1}{z_i - z_\pm} \right) \]

Apply reflection identity for Spence function:

\[ S_p(x) + \ln(x) \ln(1-x) = -S_p(1-x) + \frac{1}{6} \pi^2 \]

(valid if \( x \notin (-\infty, 0] \cup [1, \infty) \), but \( 2\pi \) prescription keeps \( x \) off the real axis.)

\[ R_\pm = -L_{i_2} \left( \frac{1 - z_i}{z_i - z_\pm} \right) + L_{i_2} \left( \frac{-z_i}{z_i - z_\pm} \right) + \ln \left( \frac{1 - z_i}{z_i - z_\pm} \right) \eta \left( 1 - z_\pm, \frac{1}{z_i - z_\pm} \right) - \ln \left( \frac{-z_i}{z_i - z_\pm} \right) \eta \left( - z_\pm, \frac{1}{z_i - z_\pm} \right) \]
Therefore, we have:

\[ I_i = \sum_{\pm} R_{i,\pm} \]

\[
C_0(p_i^2, q_i^2, p_{2i}^2; m_0, m_{\pm}, m_i) = -\frac{1}{\epsilon + 2b_e} \left[ I_{2i} + I_{12i} + I_{2} \right]
\]

\[
= -\frac{1}{\sqrt{\lambda(q_i^2, p_i^2, p_{2i}^2)}} \sum_{i = \{1, 2, 3\}} I_{(i)}
\]

\[
= -\frac{1}{\sqrt{\lambda(q_i^2, p_i^2, p_{2i}^2)}} \sum_{i = \{1, 2, 3\}} \sum_{\pm} R_{(i),\pm}
\]

\[
= \frac{1}{\sqrt{\lambda(q_i^2, p_i^2, p_{2i}^2)}} \sum_{i = \{1, 2, 3\}} \sum_{\pm} \left[ L_{i,2} \left( \frac{1 - Z_i}{Z_{i,\pm} - Z_i} \right) - L_{i,2} \left( \frac{-Z_i}{Z_{i,\pm} - Z_i} \right) \right]
\]

\[
- \ln \left( \frac{1 - Z_i}{Z_{i,\pm} - Z_i} \right) \eta \left( 1 - \frac{1}{Z_{i,\pm} - Z_i} \right) + \ln \left( \frac{-Z_i}{Z_{i,\pm} - Z_i} \right) \eta \left(-\frac{1}{Z_{i,\pm} - Z_i} \right)
\]
Roots of polynomials:

\[ Z_{1\pm} = \frac{1}{a} \left[ (1 + \frac{m_1^2 - m_0^2}{p_1^2}) \pm \frac{1}{p_1^2} \sqrt{\lambda(p_1^2, m_0^2, m_1^2) + 4p_1^2 i\varepsilon} \right] \]

\[ Z_{2\pm} = \frac{1}{a} \left[ (1 + \frac{m_0^2 - m_2^2}{p_2^2}) \pm \frac{1}{p_2^2} \sqrt{\lambda(p_2^2, m_0^2, m_2^2) + 4p_2^2 i\varepsilon} \right] \]

\[ Z_{12\pm} = \frac{1}{a} \left[ (1 + \frac{m_2^2 - m_1^2}{q_{12}^2}) \pm \frac{1}{q_{12}^2} \sqrt{\lambda(q_{12}^2, m_1^2, m_2^2) + 4q_{12}^2 i\varepsilon} \right] \]

Position of pseudo-poles:

\[ Z_o = \frac{-1}{2p_2^2 \lambda(q_{12}, m_2^2)} \left[ 2p_2^2 (-p_2^2 + m_2^2 - m_0^2) + 2p_2^2 (p_1^2 + q_{12}^2 + m_0^2 + m_2^2) + \lambda^{\frac{1}{2}}(p_2^2 - q_{12}^2 - p_0^2 + m_2^2 \right] \]

\[ Z_1 = \frac{1}{2} \left[ (1 + \frac{m_1^2 - m_0^2}{p_1^2}) + \frac{(q_{12}^2)^2 - p_1^2(q_{12}^2 + q_{12}^2 + m_0^2 - m_1^2) + (p_2^2 - q_{12}^2)(m_0^2 - m_1^2)}{p_1^2 \lambda^{\frac{1}{2}}(p_1^2, p_0^2, p_2^2)} \right] \]

\[ Z_2 = \frac{1}{2} \left[ (1 + \frac{m_0^2 - m_2^2}{p_2^2}) + \frac{(q_{12}^2)^2 - p_2^2(q_{12}^2 + q_{12}^2 + m_2^2 - m_0^2) + (p_1^2 - q_{12}^2)(m_0^2 - m_2^2)}{p_2^2 \lambda^{\frac{1}{2}}(q_{12}, p_0^2, p_2^2)} \right] \]

\[ Z_{12} = \frac{1}{2} \left[ (1 + \frac{m_2^2 - m_1^2}{q_{12}^2}) + \frac{(q_{12}^2)^2 - q_{12}^2(q_{12}^2 + q_{12}^2 + m_0^2 - m_1^2) + (p_2^2 - q_{12}^2)(m_0^2 - m_2^2)}{q_{12}^2 \lambda^{\frac{1}{2}}(q_{12}, p_0^2, p_2^2)} \right] \]

For general channel: \( i = 1, 2, 12 \)

\[ Z_{1\pm} = \frac{1}{2} \left[ (1 + \frac{m_1^2 - m_0^2}{p_{1i}^2}) \pm \frac{1}{p_{1i}^2} \sqrt{\lambda(p_{1i}^2, m_0^2, m_1^2) + 4p_{2i}^2 i\varepsilon} \right] \]

\[ Z_i = \frac{1}{2} \left[ (1 + \frac{m_i^2 - m_0^2}{p_i^2}) + \frac{(p_{1i}^2)^2 - p_1^2(p_{1i}^2 + p_0^2 + m_0^2 - 2m_i^2) + (p_2^2 - p_{1i}^2)(m_i^2 - m_0^2)}{p_i^2 \lambda^{\frac{1}{2}}(p_{1i}, p_0^2, p_2^2)} \right] \]

\[
\{ p_{0,1,2}, p_{0,1,2}; m_0, m_0, m_0 \}
\]

\[ = \{ p_{1i}^2, p_{1i}^2, p_{1i}^2; m_0^2, m_0^2, m_0^2 \} \quad \text{if} \quad i = 1 \]

\[ \{ p_{1i}^2, q_{12}^2, p_{1i}^2; m_0^2, m_0^2, m_0^2 \} \quad \text{if} \quad i = 2 \]

\[ \{ q_{12}^2, p_{1i}^2, p_{1i}^2; m_0^2, m_0^2, m_0^2 \} \quad \text{if} \quad i = 12 \]
With perseverance, can show that arguments of dilogs are as follows:

\[
1 - z_i = 1 + \frac{\lambda_2(p_1^2, p_2^2, q_2^2)}{\lambda_2(p_1^2, p_2^2, q_2^2)} \left[ \frac{p_1^2 - m_b^2 + m_c^2}{m_b^2 - m_c^2} \right] \lambda_2(p_1^2, p_2^2, q_2^2)
\]

\[
Z_i - Z_\pm = -1 + \frac{\lambda_2(p_1^2, p_2^2, q_2^2)}{\lambda_2(p_1^2, p_2^2, q_2^2)} \left[ \frac{p_1^2 - m_b^2 + m_c^2}{m_b^2 - m_c^2} \right] \lambda_2(p_1^2, p_2^2, q_2^2)
\]

\[
Z_i - Z_\pm = -1 + \frac{\lambda_2(p_1^2, p_2^2, q_2^2)}{\lambda_2(p_1^2, p_2^2, q_2^2)} \left[ \frac{p_1^2 - m_b^2 + m_c^2}{m_b^2 - m_c^2} \right] \lambda_2(p_1^2, p_2^2, q_2^2)
\]

where channels:

\[
\{ p_1^2, p_2^2, q_2^2 ; m_1^2, m_2^2, m_3^2 \}
\]

\[
\begin{cases}
\{ p_1^2, p_2^2, q_2^2 ; m_1^2, m_2^2, m_3^2 \} & \text{if } i = 1 \\
\{ q_2^2, p_1^2, p_2^2 ; m_2^2, m_3^2, m_1^2 \} & \text{if } i = 2 \\
\{ p_2^2, q_2^2, p_1^2 ; m_3^2, m_1^2, m_2^2 \} & \text{if } i = 3
\end{cases}
\]

Then

\[
C_0(p_1^2, q_2^2, p_2^2 ; m_1, m_2, m_3)
\]

\[
= \frac{1}{\lambda_2(p_1^2, p_2^2, q_2^2)} \sum_{i=1,2,3} \sum_{j=1,2} \left[ \text{Li}_2 \left( \frac{1 - z_i}{Z_i - Z_\pm} \right) - \text{Li}_2 \left( \frac{-z_i}{Z_i - Z_\pm} \right) \right]
\]

\[
- \ln \left( \frac{1 - z_i}{Z_i - Z_\pm} \right) \eta \left( \frac{1}{Z_i - Z_\pm} \right) + \ln \left( \frac{-z_i}{Z_i - Z_\pm} \right) \eta \left( \frac{1}{Z_i - Z_\pm} \right)
\]

\[
- \ln \left( \frac{1 - z_i}{Z_i} \right) \left[ \eta \left(-z_i, -z_i\right) - \eta \left(z_i - z_i, z_i - z_i\right) + \eta \left(\alpha - i\epsilon, \alpha - i\epsilon\right) \right]
\]

Final line needed if $\lambda(p_1^2, p_2^2, q_2^2) \leq 0$. 
Simplifying the prescription for analytic expressions

Assuming \( \lambda(p^2, p^2, p^2) \geq 0 \) \( \Rightarrow \) all \( \zeta_i \) are real-valued.

Recall that \( \zeta_{\pm} \) are solutions to:

\[
P_a^2 \zeta^2 - (p_a^2 - m_1^2 + m_2^2) \zeta + m_3^2 - i \epsilon = 0
\]

\[
P_b^2 \zeta^2 - (p_b^2 - m_1^2 + m_3^2) \zeta + m_2^2 - i \epsilon = 0
\]

\[
P_c^2 \zeta^2 - (p_c^2 - m_2^2 + m_3^2) \zeta + m_1^2 - i \epsilon = 0
\]

\[
\mathcal{C}(\epsilon) : \quad P(\zeta^{(0)}) = 0
\]

\[
\mathcal{C}(\epsilon^2) : \quad z^{(2)}_{\pm} = \frac{1}{P'(\zeta^{(2)})} = \pm \sqrt{\lambda(p_a^2, p_b^2, p_c^2)}
\]

Then, the arguments of Dilogarithms are:

\[
\frac{1 - z_i}{z_d - z_i} \approx \frac{1 - z_i}{z_d^{(2)} - z_i} + \frac{1}{(z_d^{(2)} - z_i)^2} z_{\pm}^{(2)} (-1 + z_i) i \epsilon + \ldots
\]

\[
\frac{-z_i}{z_d - z_i} \approx \frac{-z_i}{z_d^{(2)} - z_i} + \frac{1}{(z_d^{(2)} - z_i)^2} z_{\pm}^{(2)} z_i i \epsilon + \ldots
\]

always positive

\[\frac{1}{P'(z^{(2)})}\] are always oppositely signed.

Thus, the dilogs read:

\[
\text{Li}_2 \left( \frac{1 - z_i}{z_d^{(0)} - z_i} \pm (-1 + z_i) i \epsilon \right) \quad \text{&} \quad \text{Li}_2 \left( \frac{z_i}{z_d^{(0)} - z_i} \pm z_i i \epsilon \right)
\]