Quantum Chromodynamics — Formulation

- Based on group $SU(3)$
  
  Fermions: Quarks (fundamental rep.) $i = 1, 2, 3$
  
  Gauge bosons: Gluons (adjoint rep.) $a = 1, 2, \ldots, 8$

- Interact via exchanging color, the "charge" of QCD.

- Six (known) quark flavors (up, down, charm, strange, top, bottom) with different "masses."

Lagrangian:

$$L_{QCD} = \sum_f \bar{\psi}_f \left( i \gamma^\mu \left( \gamma^5 \right) \right) \gamma_5 \gamma^\mu \psi_f - \frac{1}{4} G_{\mu
u}^a G^{\mu
u a} + \frac{g^2}{16\pi^2} \theta_{QCD} \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^a G_{\rho\sigma}^a$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c \equiv \text{field strength tensor.}$$

(can be written $G_{\mu\nu}^a T^a = \frac{i}{g} [D_\mu, D_\nu]$)

$$D_\mu \psi = \left( \partial_\mu + ig T^a A_\mu^a \right) \psi \equiv \text{covariant derivative.}$$

$$A_\mu^a \equiv \text{gauge fields}$$

$$g = \text{strong coupling constant} \quad \text{— sometimes written } g_3$$

Let define $\alpha_s = g^2/4\pi$

often convenient to extract another factor of $\pi$:

$$a = \frac{\alpha_s}{\pi}$$

Like any typical quantum field theory, all of these quantities will have to be renormalized in some particular scheme (usually $\overline{\text{MS}}$).
The generators of $SU(3)$, $T^a$, satisfy the $SU(3)$ Lie algebra:

$$[T^a, T^b] = i f^{abc} T^c$$

Structure constants (totally antisymmetric)

In the standard representation, the generators, $T^a$, are represented by Gell-Mann matrices, $\lambda^a$:

$$T^a = \frac{1}{2} \lambda^a$$

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
\lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
\lambda_9 &= \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{align*}
\]

All hermitian and traceless. Two are diagonal, $\lambda_3$ & $\lambda_8$, and commute with each other $\Rightarrow$ rank of $SU(3)$ is 2.

Structure constants in this representation:

$$f^{abc} = -2 i \text{ Tr}([T^a, T^b] T^c)$$

$$f_{323} = 1, \quad f_{358} = f_{678} = \sqrt{3}/2$$

$$f_{147} = f_{158} = f_{236} = f_{257} = f_{345} = f_{376} = \frac{1}{2}.$$
Covariant Derivative:

Always \((D_{\mu})_{ij} = \delta_{ij} \partial_{\mu} + i g T^{a}_{ij} A_{\mu}^{a} \) \(\leftarrow\) any representation \( T^{a}_{ij} \)

acting on an adjoint index \((\text{change } i,j \rightarrow a,b \text{ rename } a \rightarrow c)\)

\((D_{\mu})^{ab} = \delta^{ab} \partial_{\mu} + i g (T^{c})^{ab} A_{\mu}^{c} \quad \text{but } (T^{c})^{ab} = -i f^{cab} \)

\[= \delta^{ab} \partial_{\mu} + g f^{cab} A_{\mu}^{c} \]

\[= \delta^{ab} \partial_{\mu} + g f^{abc} A_{\mu}^{c} \]

Commutator:

\([D_{\mu}, D_{\nu}] = [\partial_{\mu} + i g T^{a} A_{\mu}^{a}, \partial_{\nu} + i g T^{b} A_{\nu}^{b}]\)

\[= i g T^{c} (\partial_{\mu} A_{\nu}^{c} - \partial_{\nu} A_{\mu}^{c}) - g^{2} A_{\mu}^{a} A_{\nu}^{b} [T^{a}, T^{b}] \]

(from abelian case)

\[= i g T^{c} (\partial_{\mu} A_{\nu}^{c} - \partial_{\nu} A_{\mu}^{c}) - g^{2} A_{\mu}^{a} A_{\nu}^{b} i f^{abc} T^{c} \]

\[= i g T^{c} \left[ (\partial_{\mu} A_{\nu}^{c} - \partial_{\nu} A_{\mu}^{c}) - g f^{abc} A_{\mu}^{a} A_{\nu}^{b} \right] \]

\[= i g T^{c} F_{\mu \nu}^{c} \]

\([\text{when dealing with matrix-valued fields, one defines } i g T^{c} F_{\mu \nu}^{c} = g f^{abc} F_{\mu \nu}^{c} = (F_{\mu \nu})^{ab} \text{ (in adj. rep)}\]