Calculating the residue of pole, \(Z_{\text{res}}\), given by:

\[
-i \Sigma(\not{p}) = -\bigotimes_{\not{p}} = \sum_{\not{p}}^\gamma + \bigotimes_{\not{p}}
\]

In Feynman gauge, \(g=1\), this is (\ref{eq:1})

\[
= \frac{eg^2C_F}{(4\pi)^2} \left[ -\not{p} \left( 1 + \int_0^1 dx \frac{2(1-x)\ln \left( \frac{-x(1-x)p^2 + x m^2}{\mu^2} \right)}{2(1-x)p^2 + x m^2} \right) \right.
\]

\[
+ m \left( 2 + 4 \int_0^1 dx \frac{1}{1-x} \ln \left( \frac{-x(1-x)p^2 + x m^2}{\mu^2} \right) \right] \right)
\]

Notice: For massless quarks, \(m=0\), taking derivatives and setting \(\not{p} = m = 0\) causes the log to diverge \(\ln(0)\) - undefined. This is due to an IR singularity.

- Back up: use expression for general \(d\):

\[
-i \Sigma(\not{p}) = \frac{eg^2C_F}{(4\pi)^2} \int_0^1 dx \left\{ \left( d-2 \right) (1-x) \not{p} \right\} \mu^{2d} \Gamma \left( \frac{2}{d-2} \right) \left( \frac{1}{-x(1-x)p^2} \right)^{\frac{d}{2}-1} + i \delta_{d,4} - \delta_{d,4} m
\]

To obtain the counter-term, we needed the UV singularity. We took \(d=4-2\varepsilon\), with \(\varepsilon\) positive (less than 4 dim.) and found the \(\frac{1}{\varepsilon}\) pole. Having fixed the counterterm, the diagram is free of UV divergences - we may now take \(\varepsilon < 0\) (more than 4 dim) to extract the IR singularity when \(m=0\), and when \(\not{p}\) is set to \(m=0\).

\[
-i \frac{d}{d\not{p}} \Sigma(\not{p}) = \frac{eg^2C_F}{(4\pi)^2} \int_0^1 dx \left\{ \left( d-2 \right) (1-x) \not{p} \right\} \mu^{2d} \Gamma \left( \frac{2}{d-2} \right) \not{p} \left( d-4 \right) \frac{1}{x(1-x)} \frac{1}{p^2} \right)
\]

\[
+ \left( d-2 \right) (1-x) \not{p} \right\} \mu^{2d} \Gamma \left( \frac{2}{d-2} \right) \not{p} \left( d-4 \right) \frac{1}{x(1-x)} \frac{1}{p^2} \right)
\]

Combine

\[
= \frac{eg^2C_F}{(4\pi)^2} \int_0^1 dx \left( 1 + \frac{d-4}{2} \right) \left( d-2 \right) (1-x) \mu^{2d} \Gamma \left( \frac{2}{d-2} \right) \left( \frac{1}{x(1-x)p^2} \right)^{\frac{d}{2}-1} + i \delta_y
\]

Now, setting \(p^2 = m^2 = 0\), since the exponent \(2 - \frac{d}{2} = +\varepsilon < 0\), the \(-x(1-x)p^2\) is effectively in the numerator, and the integral vanishes.
We're left with

\[-i \frac{d \Sigma(p)}{dp} \bigg|_{p^2 = m^2} = i \delta p \quad \Rightarrow \quad \frac{d \Sigma(p)}{dp} \bigg|_{p^2 = m^2} = -\delta p\]

\[\frac{g^2}{(4\pi)^2} (-C_F) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right) \quad \text{this is the IR divergent part.}\]

What's going on? In the massless, on-shell limit, the self-energy diagram vanishes since the dimensionally regulated integral has no scale — the UV divergent and IR divergent parts of the same diagram exactly cancel.

Essentially:

\[-i \Sigma(p) = \left( \text{UV div.} \right) + \left( \text{IR div.} \right) + \left( \delta \text{C.T. uv. div.} \right)\]

\[\text{cancel} \uparrow \quad \text{designed to cancel} \quad \Rightarrow \left( \text{IR div.} \right) = \left( \delta \text{C.T. uv. div.} \right)\]

Hence, the residue of the quark pole is IR divergent:

\[Z_{rs} = \left( 1 + \frac{g^2}{(4\pi)^2} (-C_F) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right) \right)^{-1} = (1 + \delta p)^{-1}\]

\[\approx 1 - \frac{g^2}{(4\pi)^2} (-C_F) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right) = 1 - \delta p\]

So, now the matrix element for $e^+e^- \rightarrow q\bar{q}$ at NLO (virtual correction) is:

\[Z_{\text{res}} \times \left( \frac{\gamma}{2} + \frac{\gamma}{2} + \frac{\gamma}{2} \right) = (1 - \delta p) \left( \frac{\gamma}{2} + \frac{\gamma}{2} + \frac{\gamma}{2} \right)\]

\[= \frac{\gamma}{2} (1 - \delta p) + \frac{\gamma}{2} + \frac{\gamma}{2} \quad \text{But} \quad \frac{\gamma}{2} = \frac{\gamma}{2} \delta q_{44}\]

\[= \frac{\gamma}{2} (1 - \delta p + \delta q_{44}) + \frac{\gamma}{2}\]

\[\text{cancel since} \quad Z_{\gamma} = Z_{\gamma_{44}} \text{ in QED}\]

\[= \frac{\gamma}{2} + \frac{\gamma}{2} . \text{ All UV div. gone} \Rightarrow \text{left with IR div.}\]