Spin from Relativity - the Dirac equation.

**History:**

At Solvay conference (Oct 1927) Bohr's conversation w/ Dirac:

Bohr: What are you working on?

Dirac: I'm trying to get a relativistic theory of the electron.

Bohr: But Klein has already solved that problem.

Dirac did not agree. The Klein-Gordon wave equation does not give a positive-definite \( \rho \). Can't interpret as probability density.

**Dirac's construction**

Dirac wanted a linear equation in \( \frac{\partial}{\partial t} \)

(reason: solution then requires only one boundary condition (just like Schrödinger equation).

But linear in \( \frac{\partial}{\partial t} \) and relativistically covariant \( \Rightarrow \) MUST be linear in \( \frac{\partial}{\partial x} \):

**Dirac tries:**

\[
i \hbar \frac{\partial}{\partial t} \psi = \left( c \hat{\alpha} \cdot \hat{p} + \beta mc^2 \right) \psi
\]

Since \( H = E \) \( \Rightarrow \) want \( \hat{H}^2 = \hat{p}^2 c^2 + m^2 c^4 \) (Einstein's relationship).

Problem: If \( \hat{\alpha} \) and \( \beta \) are treated as numbers,

\[
\hat{H}^2 = c^2 (\hat{\alpha} \cdot \hat{p})^2 + \beta^2 m^2 c^4 + 2 (\hat{\beta} \hat{\alpha}) \cdot \hat{p} \cdot m c^3
\]

Need:

\( \alpha_i \alpha_j = \delta_{ij} \) \( \imp \) No solutions!

\( \beta^2 = 1 \) \( \imp \beta = 1 \)

\( \beta \alpha_i = 0 \) \( \imp \alpha_i = 0 \) ?!

Doesn't work.
Obviously, $\alpha$ & $\beta$ can't be commuting numbers

must be algebraic matrices of some kind... try matrices.

$$i\hbar \frac{\partial}{\partial t} \psi = (c \alpha_i \hat{p}_i + \beta mc^2) \psi \quad \alpha_1, \alpha_2, \alpha_3, \beta \equiv \text{matrices.}$$

$\psi \equiv \text{column vector.}$

Then $\hat{A}^2$ yields:

$$\hat{A}^2 = c^2 \alpha_i \alpha_j \hat{p}_i \hat{p}_j + \beta^2 m^2 c^4 + (\alpha_i \beta + \beta \alpha_i) \hat{p}_i \hat{p}_i mc^3$$

```plaintext
\alpha_i \alpha_j + \alpha_j \alpha_i \\
\beta^2 = 1 \\
\alpha_i \beta + \beta \alpha_i = 0
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match to $\hat{A}^2 = \gamma c^2 + m^2 c^4$

$$\left[ \begin{array}{c} \frac{1}{2} (\alpha_i \alpha_j + \alpha_j \alpha_i) = \delta_{ij} \\ \beta^2 = 1 \\ \alpha_i \beta + \beta \alpha_i = 0 \end{array} \right]$$

for all $i, j \in \{1, 2, 3\}$

In 1+1 dimensions,
we only have one $\alpha$ and $\beta$

solution: $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

\[ \sigma_1 \quad \text{and} \quad \sigma_2 \]

In 2+1 dimensions,
we have two $\alpha_i$: ($\alpha_1 \& \alpha_2$) and a $\beta$:

solution: $\alpha_1 = \begin{pmatrix} 1 & 1 \\ \sigma_1 \end{pmatrix}$ $\alpha_2 = \begin{pmatrix} 1 & -1 \\ \sigma_2 \end{pmatrix}$ $\beta = \begin{pmatrix} 1 & 1 \\ \sigma_3 \end{pmatrix}$

$\sigma_2$ is an extra matrix that serves as parity.
In 3+1 dimensions, there aren't enough 2x2 matrices — must go to 4x4 matrices.

Dirac $\alpha$ & $\beta$ matrices

$$\begin{align*}
\alpha_1 &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \alpha_2 &= \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} & \alpha_3 &= \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} & \beta &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}$$

(Modern) Dirac $\gamma$ matrices: covariant form of Dirac equation

$$i\hbar \frac{\partial}{\partial t} \psi = (c \vec{\alpha} \cdot \vec{p} + \beta mc^2) \psi$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Multiply on left by $\beta$:

$$i\beta \frac{\partial}{\partial t} \psi = (c \beta \vec{\alpha} \cdot \vec{p} + \beta^2 mc^2) \psi$$

In coordinate rep $\vec{p} \equiv -i\hbar \nabla$

$$i\hbar \beta \frac{\partial}{\partial t} \psi = \left[ -i\hbar c (\beta \vec{\alpha}) \cdot \nabla + \frac{1}{2} mc^2 \right] \psi$$

$$\left[ i\hbar \beta \frac{\partial}{\partial t} + i\hbar c (\beta \vec{\alpha}) \cdot \nabla - \frac{1}{2} mc^2 \right] \psi = 0$$

Divide by $\hbar c$

$$\left[ i\beta \frac{1}{c} \frac{\partial}{\partial t} + i(\beta \vec{\alpha}) \cdot \nabla - \frac{1}{2} \frac{mc}{\hbar} \right] \psi = 0$$

Abbreviate: $\gamma^0 = \beta$, $\vec{\gamma} = \beta \vec{\alpha}$

$$\left[ i \gamma^0 \frac{1}{c} \frac{\partial}{\partial t} + i \vec{\gamma} \cdot \nabla - \frac{mc}{\hbar} \right] \psi = 0$$

$$\left[ i \gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right] \psi = 0 \quad \text{covariant Dirac equation.}$$