Charged particle in a homogeneous magnetic field

Orient \( z \)-axis along \( \vec{B} \)-field.

Need vector potential \( \vec{A} \), such that \( \vec{B} = \nabla \times \vec{A} \).

- Many choices/gauges

Symmetric gauge: \( \vec{A} = -\frac{1}{2} \vec{x} \times \vec{B} \).

- Satisfies Coulomb gauge condition:
  \[
  \nabla \cdot \vec{A} = -\frac{1}{2} \nabla \cdot (\vec{x} \times \vec{B}) = \frac{1}{2} \epsilon_{ijk} \nabla_i (x_j B_k) = -\frac{1}{2} \epsilon_{ijk} \delta_{ij} B_k = 0 \quad \checkmark
  \]

- Indeed gives \( \vec{B} = \nabla \times \vec{A} \)

  \[
  \nabla \times \vec{A} = -\frac{1}{2} \nabla \times (\vec{x} \times \vec{B}) = \frac{1}{2} \epsilon_{ijk} \vec{e}_{k m} \left( \nabla_j x_k \right) B_m
  \]

  \[
  = \frac{1}{2} \epsilon_{ijk} \epsilon_{k m} \left( \nabla_j x_k \right) B_m
  \]

  \[
  = \frac{1}{2} \left( \delta_{ij} \delta_{jm} - \epsilon_{im} \times 3 \right) B_m = \vec{B} \quad \checkmark
  \]

Component form:

\[
\vec{A} = -\frac{1}{2} \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
x & y & z \\
0 & 0 & |B|
\end{vmatrix} = -\frac{\vec{B}}{2} (y \hat{i} - x \hat{j})
\]

Axial gauge: \( \vec{A} = -B y \hat{i} \quad or \quad +B x \hat{j} \)

preserves translations along \( x \)-axis \quad \text{along \( y \)-axis}
Then Schrödinger equation in homogeneous $B$-field becomes:

\[
\left( -\frac{\hbar^2}{2m} \nabla^2 - \frac{\imath e}{2m} (\mathbf{x} \times \mathbf{B}) \cdot \nabla + \frac{\imath e}{2m} (\nabla \times \mathbf{A}) - \frac{\hbar^2 \imath}{2m} \mathbf{B} \cdot (\nabla \times \mathbf{A}) \right) \psi = \varepsilon \psi
\]

\[
= \varepsilon \left[ \frac{\hbar^2}{2m} \nabla^2 - \frac{\imath e}{2m} \mathbf{B} \cdot \nabla + \frac{\hbar^2 \imath}{2m} \mathbf{B} \cdot \nabla \right] \psi
\]

\[
= \varepsilon \left[ \left( \mathbf{B}^2 - (\mathbf{x} \cdot \mathbf{B})^2 \right) \nabla^2 - \frac{\hbar^2 \imath}{2m} \mathbf{B} \cdot \nabla \right] \psi
\]

\[
= \varepsilon \mathbf{B}^2 \psi - \frac{\hbar^2 \imath}{2m} \mathbf{B} \cdot \nabla \psi
\]

so

\[
\left( -\frac{\hbar^2}{2m} \nabla^2 - \frac{\hbar^2 \imath}{2m} \mathbf{B} \cdot \nabla \right) \psi = \varepsilon \psi
\]

\[
\left( -\frac{\hbar^2}{2m} \nabla^2 - \frac{\hbar^2 \imath}{2m} \mathbf{B} \cdot \nabla \right) \psi = \varepsilon \psi
\]

\[
\left[ \mathbf{B}^2 - (\mathbf{x} \cdot \mathbf{B})^2 \right] \nabla^2 \psi + \left( \frac{\hbar^2 \imath}{2m} \mathbf{B} \cdot \nabla \right) \psi = \varepsilon \psi
\]

\[
\left( \frac{\hbar^2}{8m} [x^2 \mathbf{B}^2 - (\mathbf{x} \cdot \mathbf{B})^2] - g \frac{\hbar}{2m} \mathbf{B} \cdot \mathbf{B} \right) \psi(x) = \mathcal{E} \psi(x)
\]

\[
\text{Symmetric gauge}
\]

\[
\text{contributes to paramagnetism}
\]

\[
\text{contributes to diamagnetism}
\]

\[
\text{contributes to ferromagnetism}
\]

\[
\text{orbital motion of positive charges}
\]

\[
\text{Vector potential in symm. gauge}
\]

\[
\text{magnetic field}
\]
Motion of charged particle:

Assumed $\mathbf{B} = (0, 0, B)$. Then Schrödinger equation becomes

$$\left( \frac{\hbar^2}{2m} \left( p_x^2 + p_y^2 \right) + \frac{e}{2m} B L_z + \frac{e^2 B^2}{8m} \left( \frac{1}{(x^2 + y^2) B^2} \right) \right) \Psi = E \Psi$$

Separate variables in $x, y$ & $z$ (spherical)
Put $\Psi(x) = f(x, y) \phi(z) \chi_{\text{spin}}$.

$$\frac{\hbar^2}{2m} \left( p_x^2 + p_y^2 \right) f(x, y) + f \chi \frac{1}{2m} \left( \frac{e}{m} B L_z f(x, y) \right) + \phi \chi \frac{e^2 B^2}{8m} \left( \frac{1}{x^2 + y^2} \right) f(x, y) = E \phi \chi$$

Divide by $\phi \chi = f \phi \chi$.

$$\frac{1}{f(x, y)} \left[ \frac{1}{2m} \left( p_x^2 + p_y^2 \right) - \frac{e}{2m} B L_z + \frac{e^2 B^2}{8m} \left( \frac{1}{x^2 + y^2} \right) \right] f(x, y)$$

$$+ \frac{1}{\phi(z)} \left[ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \phi(z) \right] + \chi_{\text{spin}} \left[ -g \frac{e}{2m} B S_z \chi_{\text{spin}} \right] = E$$

Free particle motion along $z$-direction:
Pauli spin in B-field:

$$\phi(z) \sim e^{i k_z z}$$

$$\chi_{\pm} \sim \begin{cases} 1 & s_z = \pm \frac{1}{2} \\ 0 & s_z = \mp \frac{1}{2} \end{cases}$$

Separation constant:
Separation constant:

$$E_z = \frac{-\hbar^2 k_z^2}{2m}$$

$$E_{\text{spin}} = -\frac{\hbar}{2m} \frac{e}{B} m_s$$

Solve S.E. for $x, y$ part (Hard!): Define separation constant $= E_B$

$$\therefore E = E_B + E_z + E_{\text{spin}}$$
Motion in x-y plane: Classical considerations

\[ x^2 + y^2 = r^2 \]

**Magnitudes:**

\[ \frac{mv^2}{r} = evB = mw^2r \]

\[ r = \frac{mv}{eB} = \text{Cyclotron radius} \]

\[ \omega = \frac{eB}{m} = \text{Cyclotron frequency} \]

\[ E = \frac{1}{2}mv^2 = \frac{m}{2}r^2\omega^2 = \text{Cyclotron energy} \]

**Vector relations:**

Lorentz force law:

\[ \vec{F} = e \vec{v} \times \vec{B} = m\vec{a} \]

\[ m\vec{a} = e \begin{vmatrix} i & j & k \\ u_x & u_y & u_z \\ 0 & 0 & B \end{vmatrix} = B(\omega_y \hat{i} - \omega_x \hat{j}) \]

**Accelerations (EDU) 1st integral (velocities) 2nd integral (positions)**

\[ \ddot{x} = B\omega_y \Rightarrow \dot{v}_x = -\omega r \sin \omega t \quad x = X + r \cos \omega t \]

\[ \ddot{y} = B\omega_x \Rightarrow \dot{v}_y = -\omega r \cos \omega t \quad y = Y - r \sin \omega t \]

\[ \omega, \omega \text{- integration const.} \]

\[ X, Y = \text{integration constant} \]

\[ X = X, Y = \text{center of cyclotron motion (guiding center)} \]

\[ \longrightarrow \text{constant of motion} \]

**Integration constant**

\[ \vec{X} = X\hat{i} + Y\hat{j} = \text{center of cyclotron motion (guiding center)} \]

solve 2nd integral for \( X, Y \)

\[ X = x(t) - r \cos \omega t = x(t) + \frac{v_y(t)}{\omega} \]

\[ Y = y(t) + r \sin \omega t = \frac{y(t) + v_x(t)}{\omega} \]

\[ \text{constant of motion} \]

solve 2nd integral for \( X, Y \)}