Coupling of Spin-$\frac{1}{2}$ and Orbital angular momentum

$$\vec{J} = \vec{L} + \vec{S}$$

Spin

$$S^z|\Psi_{1/2}\rangle = \hbar^2 (1/2 + 1) |\Psi_{1/2}\rangle$$

Orbital

$$L^z|\Psi_{j}\rangle = \hbar^2 \left( \frac{j(j+1)}{2} \right) |\Psi_{j}\rangle$$

$$L^z|\Psi_{j}\rangle = \pm \frac{m_L}{\hbar} |\Psi_{j}\rangle$$

TRIANGULARITY CONDITION

$$|L - S| \leq j \leq |L + S| \Rightarrow \left| L - \frac{3}{2} \right| \leq j \leq \left| L + \frac{3}{2} \right|$$

For each $L$, $j$ can take on at most 2 values:

$$j = \pm \frac{1}{2}$$

For each $j$, $L$ can take on at most 2 values:

$$L = j \pm \frac{3}{2}$$

The possible azimuthal (projection) quantum numbers are subject to

$$m_J = m_L + m_S$$

CONDITION ON PROJECTION

$S = \pm \frac{1}{2}$

Choose any $L \geq 0$. Then $j = L \pm \frac{1}{2}$. Then write down possible $m_J$.

This specifies $|L, j, m_j\rangle$.

Since $S = \pm \frac{1}{2}$, $m_S = \pm \frac{1}{2}$ meaning that there are at most 2 terms in the Clebsch-Gordan expansion.

$m_J$ fixed by

$$m_J = m_L - m_S$$

$L = 0 \rightarrow j = \frac{1}{2}$

$$m_J = \pm \frac{1}{2} : \begin{cases} m_S = \uparrow \Rightarrow m_J = 0 \text{ impossible} \\ m_S = \downarrow \Rightarrow m_J = \mp \text{ impossible} \end{cases}$$

$L = 1$

$$j = \frac{1}{2} \rightarrow j = \pm \frac{1}{2}$$

$$m_J = \pm \frac{1}{2} : \begin{cases} m_S = \uparrow \Rightarrow m_J = 0 \\ m_S = \downarrow \Rightarrow m_J = \pm 1 \text{ possible} \end{cases}$$

$$m_J = \pm \frac{1}{2} : \begin{cases} m_S = \uparrow \Rightarrow m_J = 0 \\ m_S = \downarrow \Rightarrow m_J = \mp 0 \text{ possible} \end{cases}$$

$L = 1$

$$j = \frac{3}{2} \rightarrow j = \pm \frac{3}{2}$$

$$m_J = \pm \frac{3}{2} : \begin{cases} m_S = \uparrow \Rightarrow m_J = \pm 0 \text{ possible} \\ m_S = \downarrow \Rightarrow m_J = \mp \text{ impossible} \end{cases}$$

$$m_J = \pm \frac{3}{2} : \begin{cases} m_S = \uparrow \Rightarrow m_J = \pm 0 \text{ possible} \\ m_S = \downarrow \Rightarrow m_J = \mp \text{ impossible} \end{cases}$$
\[ |l, \frac{1}{2}; j = l + \frac{1}{2}, m_j \rangle = \frac{1}{\sqrt{2l + 1}} \left( \pm \sqrt{l + m_j + \frac{1}{2}} |l, m_j + \frac{1}{2}; s = \frac{1}{2}, \uparrow \rangle + \sqrt{l + m_j - \frac{1}{2}} |l, m_j - \frac{1}{2}; s = \frac{1}{2}, \downarrow \rangle \right) \]
Each state (-) in the coupled basis is a superposition of (or must)
two states in the uncoupled basis.
In the coupled basis where $\mathbf{J}^2$ is diagonalized, $\mathbf{L} \cdot \mathbf{S}$ automatically becomes diagonalized:

Write: $\mathbf{J}^2 = (\mathbf{L} + \mathbf{S})^2$

$= \mathbf{L}^2 + \mathbf{S}^2 + 2 \mathbf{L} \cdot \mathbf{S}$

$\Rightarrow \mathbf{L} \cdot \mathbf{S} = \frac{1}{2} (\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2)$

The eigenvalues of $\mathbf{L} \cdot \mathbf{S}$ for $|l, s, j, m_j\rangle$ are:

$\mathbf{L} \cdot \mathbf{S} |l, s, j, m_j\rangle = \frac{1}{2} \hbar^2 \left( j(j+1) - l(l+1) - s(s+1) \right) \text{ (indep. of } m_j) \tag{1}$

If $s = \frac{l}{2}$ (for spin-orbit coupling in atomic systems),

then $j = l \pm \frac{1}{2}$ so that

$\mathbf{L} \cdot \mathbf{S} |l, s; j = l + \frac{1}{2}, m_j\rangle = \frac{1}{2} \hbar^2 \left( (l+\frac{1}{2})(l+\frac{3}{2}) - l(l+1) - \frac{1}{2} \left( \frac{1}{2} + 1 \right) \right)$

$= \frac{\hbar^2}{2} \left( j^2 \right)$

$\mathbf{L} \cdot \mathbf{S} |l, s; j = l - \frac{1}{2}, m_j\rangle = \frac{1}{2} \hbar^2 \left( (l-\frac{1}{2})(l+\frac{1}{2}) - l(l+1) - \frac{1}{2} \left( \frac{1}{2} + 1 \right) \right)$

$= -\frac{1}{2} \hbar^2 (j+1)$