Time-dependent pictures of quantum mechanics.

**Schrödinger Picture**

Operators time-independent:
\[ \hat{A}_s = \hat{A}(0) \]
States time-dependent satisfying:
\[ i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_s = \hat{H} |\psi(t)\rangle_s \]
Solution:
\[ |\psi(t)\rangle_s = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle_s \]

**Heisenberg Picture**

States time-independent:
\[ |\psi_H\rangle = |\psi(0)\rangle \]
Operators are time-dependent satisfying:
\[ \frac{\partial}{\partial t} \hat{A}_H(t) = i/\hbar [\hat{H}, \hat{A}(t)] + \frac{2}{\hbar} \hat{A}_H(t) \]
Solution:
\[ \hat{A}_H(t) = e^{i\hat{H}t/\hbar} \hat{A}(0) e^{-i\hat{H}t/\hbar} \]

Equivalence of time development of matrix elements:

Both pictures coincide at a common reference time \( t=0 \) in this case.

Then:
\[ \langle \Psi(t) | \hat{A}_s(t) \psi(t) \rangle_s = \langle \Psi(0) | e^{i\hat{H}t/\hbar} \hat{A}(0) e^{-i\hat{H}t/\hbar} \psi(0) \rangle = \langle \Psi_H | \hat{A}_H(t) \psi_H \rangle \]

**Interaction picture** (with time independent interaction)

If \( \hat{H} = \hat{H}_0 + \hat{H}' \)

Free Hamiltonian interaction

Then start with Heisenberg picture:

\[ \langle \psi_H | \hat{A}_H(t) \psi_H \rangle = \langle \psi_H | e^{i\hat{H}t/\hbar} \hat{A}(0) e^{-i\hat{H}t/\hbar} \psi_H \rangle \]

Insert \( (e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar}) \) at these points:
\[ = \langle \psi_H | (e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar} \hat{A}(0) e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar} \hat{A}(0) e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar} \psi_H \rangle \]
\[ = \langle \psi(t) | \hat{A}_I(t) \psi(t) \rangle \]

- Operators evolve following the free Hamiltonian \( \hat{A}_0 \)
- States evolve in a strange way — as if it were governed by the interacting Hamiltonian.
Achieve time evolution operator: $\hat{U}(t) = \hat{U}(t, 0)$
- the quantity that connects the Schrödinger and Heisenberg pictures

For time-independent Hamiltonian, $\hat{H}$:

The time evolution operator is:

$$\hat{U}(t_f, t_i) = e^{-i \hat{H}(t_f - t_i) / \hbar}$$

Interpretation:

1. Evolves a Schrödinger picture state from $t_i$ to $t_f$:

$$U(t_f, t_i) |\Psi(t_i)\rangle = |\Psi(t_f)\rangle$$

2. Evolves a Heisenberg picture state from $t_i$ to $t_f$:

$$U^\dagger(t_f, t_i) \hat{A}_H(t_i) U(t_f, t_i) = \hat{A}_H(t_f)$$

Properties:

Composition:

$$\hat{U}(t_2, t_1) \hat{U}(t_1, t_0) = \hat{U}(t_2, t_0)$$

(of course, time translation form a group)

Unitary:

$$\hat{U}(t_2, t_1) = \left[ e^{-i \hat{H}(t_2 - t_1) / \hbar} \right]^\dagger$$

$$= e^{i \hat{H}(t_2 - t_1) / \hbar} = \hat{U}^{-1}(t_2, t_1)$$

$$= \hat{U}(t_1, t_2).$$

Equivalence of Schrödinger and Heisenberg pictures:

$$\langle \Psi(t) | A_s(0) |\Psi(t)\rangle_s = \langle \Psi(0) | U^\dagger(t) A_s(0) \hat{U}(t) |\Psi(0)\rangle = \langle \Psi_H | A_H(t) |\Psi_H\rangle.$$
Reasons for introducing the interaction picture (in perturbation theory!)

1. To calculate transition, naively one would try

\[ \langle f(t_f) | i(t_i) \rangle = \langle f(t_f) | i(t_i) \rangle \]

\[ = \langle f | e^{-i\hat{H}(t_f-t_i)/\hbar} | i \rangle \]

and since \( \hat{H} = \hat{H}_0 + \hat{H}' \), perhaps try:

\[ e^{-i\hat{H}(t_f-t_i)/\hbar} = e^{-i\hat{H}_0(t_f-t_i)/\hbar} e^{-i\hat{H}'(t_f-t_i)/\hbar} \]

\[ = \langle f | e^{-i(\hat{H}_0+\hat{H}')(t_f-t_i)/\hbar} | i \rangle \]

\[ \rightarrow \]

VERY hard — the only way to do this is to use

the Baker-Campbell-Hausdorff formula

\[ e^{i\hat{H}_0 - i\hat{H}'} \frac{t}{\hbar} = e^{-i\hat{H}_0 \frac{t}{\hbar}} - i\hat{H}' \frac{t}{\hbar} \]

with intention of expanding this.

\[ \rightarrow \] could make it work using Upperam Schurz expansion.

Reason to abandon Schurz form

2. Operators evolve in the according to free Hamiltonian

\[ \Rightarrow \] operator algebra very easy.