2 → 2 kinematics

\[ \text{Com: } |\vec{r}_1| = |\vec{r}_2| \equiv |\vec{r}_3| \quad (|\vec{r}_3| = |\vec{r}_2| \text{ in general}) \]
\[ |\vec{r}_2| = |\vec{r}_3| \equiv |\vec{r}_f| \]

\[ \vec{p}_1^u = (E_1, \vec{p}_1) \quad \vec{p}_3^u = (E_2, \vec{p}_f) \]
\[ \vec{p}_2^u = (E_2, -\vec{p}_1) \quad \vec{p}_4^u = (E_4, -\vec{p}_f) \]

On-shell conditions

\[ \begin{align*}
p_1^2 &= E_1^2 - \vec{p}_1^2 = m_1^2 \\
p_2^2 &= E_2^2 - \vec{p}_2^2 = m_2^2 \\
p_3^2 &= E_3^2 - \vec{p}_3^2 = m_3^2 \\
p_4^2 &= E_4^2 - \vec{p}_4^2 = m_4^2
\end{align*} \]

Define 1st invariant

\[ s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2 \leftarrow \]
\[ = (p_3 + p_4)^2 = m_3^2 + m_4^2 + 2p_3 \cdot p_4 \]

Notice:

\[ p_1 \cdot (p_1 + p_2) = m_1^2 + p_1 \cdot p_2 \]
\[ \text{in com: } \equiv E_1 (E_1 + E_2) - \vec{p}_1 \cdot (\vec{p}_1 + \vec{p}_2) \]
\[ \Rightarrow p_1 \cdot p_2 = E_1 \sqrt{s} - m_1^2 \]

After plugging,
\[ \Rightarrow s = m_1^2 + m_2^2 + 2(E_1 \sqrt{s} - m_1^2) \]
\[ \Rightarrow E_1 = \frac{1}{2\sqrt{s}} \left(m_1^2 - m_2^2 + s\right) \]

Similarly,

\[ p_2 \cdot (p_1 + p_2) = p_1 \cdot p_2 + m_2^2 \]
\[ \text{in com: } \equiv E_2 (E_1 + E_2) - \vec{p}_2 \cdot (\vec{p}_1 + \vec{p}_2) \]
\[ \Rightarrow p_1 \cdot p_2 = E_2 \sqrt{s} - m_2^2 \]

Again, after plugging into invariant
\[ \Rightarrow s = m_1^2 + m_2^2 + 2(E_2 \sqrt{s} - m_2^2) \]
\[ \Rightarrow E_2 = \frac{1}{2\sqrt{s}} \left(-m_1^2 + m_2^2 + s\right) \]
repeat for particle 3 & 4:

\[ p_3 \cdot (p_3 + p_4) = m_3^2 + p_3 \cdot p_4 = E_3 (E_3 + E_4) - \frac{\vec{p}_3 \cdot (\vec{p}_3 + \vec{p}_4)}{\sqrt{s}} = E_3 \sqrt{s} \]

Plug into invariant

\[ S = m_3^2 + m_4^2 + 2 (E_3 \sqrt{s} - m_3^2) \]

\[ \Rightarrow E_3 = \frac{1}{2 \sqrt{s}} (m_3^2 - m_4^2 + S) \]

AND, similarly,

\[ E_4 = \frac{1}{2 \sqrt{s}} (-m_3^2 + m_4^2 + S) \]

Plug energies into on-shell conditions, and solve for \( |\vec{p}_i|^2 \)

\[ p_3^2 = \left( \frac{1}{2 \sqrt{s}} (m_3^2 - m_2^2 + S) \right)^2 - |\vec{p}_3|^2 \equiv m_1^2 \]

\[ \Rightarrow |\vec{p}_3|^2 = \frac{1}{4 s} \left( m_1^2 - m_2^2 + S \right)^2 - m_1^2 \]

\[ = \frac{1}{4 s} \left[ m_3^4 + m_4^2 + s^2 - 2 m_3^2 m_4^2 - 2 s m_3^2 - 2 s m_4^2 \right] \]

\[ \equiv \lambda(m_1^2, m_2^2, S) \]

Källén Triangle Function

\[ \lambda(a, b, c) = a^2 + b^2 + c^2 - 2 a b - 2 b c - 2 a c \]

"Kinematic Function"

\[ = \left[ c - (a + b)^2 \right] \left[ c - (a - b)^2 \right] \]

(symmetric in \( a \equiv b \equiv c \))

Similarly, \( |\vec{p}_4|^2 \)

\[ = \frac{1}{4 s} \left[ m_3^4 + m_4^2 + s^2 - 2 m_3^2 m_4^2 - 2 s m_3^2 - 2 s m_4^2 \right] \]

\[ \equiv \frac{1}{4 s} \lambda(m_3^2, m_4^2, S) \] independent of \( S \)

*Special Cases:

\[ \lambda(a, a, c) = c^2 - 4 a c \]

\[ \lambda(a, a, a) = -3 a^2 \]

Property:

\[ \lambda(a, b, c) \]

As a function of \( c \), it is a quadratic with two roots:

\[ c = \frac{(a + b)^2}{4}, \text{ if } \text{sgn}(a) = \text{sgn}(b). \]

The two roots are both of the same sign as \( a \) & \( b \).
Define 2nd invariant: \( t = (\mathbf{p}_1 - \mathbf{p}_2)^2 = m_2^2 + m_3^2 - 2p_1 \cdot p_2 \)

At \( t = 0 \) for \( \theta_{com} = 0 \)

\[
t = m_2^2 + m_3^2 - 2E_1 E_3 + 2 \mathbf{p}_1 \cdot \mathbf{p}_2
\]

\[
t = m_2^2 + m_3^2 - 2E_1 E_3 + 2 \mathbf{p}_1 \parallel \mathbf{p}_2 \cos \theta_{com}
\]

Use energy relations \( E_1 = \frac{1}{2\lambda} (m_2^2 - m_3^2 + s) \) & \( E_3 = \frac{1}{2\lambda} (m_3^2 - m_4^2 + s) \)

to eliminate \( E_1 \) & \( E_3 \).

\[
t = m_2^2 + m_3^2 - \frac{1}{\lambda} (m_2^2 - m_3^2 + s) \frac{1}{\lambda} (m_3^2 - m_4^2 + s) + 2 \mathbf{p}_1 \parallel \mathbf{p}_2 \cos \theta_{com}
\]

\[
t = m_2^2 + m_3^2 - \frac{1}{\lambda} (m_2^2 - m_3^2 + s)(m_3^2 - m_4^2 + s) + 2 \mathbf{p}_1 \parallel \mathbf{p}_2 \cos \theta_{com}
\]

\[
t = \frac{4}{2\lambda} \left[ \frac{2E_1}{2\lambda} m_2^2 + 2E_3}{2\lambda} m_3^2 - (m_2^2 - m_3^2)(m_3^2 - m_4^2) \right] - s m_2^2 + s m_3^2 - s m_4^2 + s^2
\]

\[
+ 2 \mathbf{p}_1 \parallel \mathbf{p}_2 \cos \theta_{com}
\]

\[
t = \frac{4}{2\lambda} \left[ -s^2 + s \sum_{i} m_i^2 - (m_2^2 - m_3^2)(m_3^2 - m_4^2) \right] + 2 \mathbf{p}_1 \parallel \mathbf{p}_2 \cos \theta_{com}
\]

\[
\Rightarrow \cos \theta_{com} = \frac{2st + [s^2 - s \sum_{i} m_i^2 + (m_2^2 - m_3^2)(m_3^2 - m_4^2)]}{4s |\mathbf{p}_1 \parallel \mathbf{p}_2|}
\]

Then use \( |\mathbf{p}_1| = \frac{1}{2\lambda} \sqrt{\frac{1}{2}} \lambda^{\frac{1}{2}} (m_2^2, m_3^2, s) \) & \( |\mathbf{p}_2| = \frac{1}{2\lambda} \sqrt{\frac{1}{2}} \lambda^{\frac{1}{2}} (m_3^2, m_4^2, s) \)

to finally get

\[
\cos \theta_{com} = \frac{s^2 + s(2t - \sum_{i} m_i^2) + (m_2^2 - m_3^2)(m_3^2 - m_4^2)}{\lambda^{\frac{1}{2}} (m_2^2, m_3^2, s) \lambda^{\frac{1}{2}} (m_3^2, m_4^2, s)}
\]
**SPECIAL CASE:** \( A + B \rightarrow A + B \) (elastic)

In the case \( m_1 = m_3 \) and \( m_2 = m_4 \),

\[
t = \frac{1}{2s} \left[ -s^2 + s \sum \frac{m_i^2}{s_i} - (m_1^2 - m_2^2)(m_3^2 - m_4^2) \right] + 2 \left| \bar{p}_2 \right| \left| \bar{p}_3 \right| \cos \theta_{\text{com}}
\]

\[
= \frac{1}{2s} \left[ -s^2 + s(2m_1^2 + 2m_2^2) - (m_1^2 - m_2^2)^2 \right]
\]
\[
+ 2 \frac{1}{2s} \lambda^{\frac{1}{2}}(m_1^2, m_2^2, s) \frac{1}{2s} \lambda^{\frac{1}{2}}(m_1^2, m_2^2, s) \cos \theta_{\text{com}}
\]

\[
= \frac{-1}{2s} \left( s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right) + \frac{1}{2s} \lambda(m_1^2, m_2^2, s) \cos \theta_{\text{com}}
\]

but \( \lambda(m_1^2, m_2^2, s) = m_1^4 + m_2^4 + s^2 - 2m_1^2m_2^2 - 2m_1s - 2m_2s \)
\[
= s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2
\]

\[
\therefore t = \frac{-1}{2s} \lambda(m_1^2, m_2^2, s) \left( 1 - \cos \theta_{\text{com}} \right) = -2 \left| \bar{p}_1 \right| \left( 1 - \cos \theta \right) \Rightarrow \cos \theta = \frac{t}{2 \left| \bar{p}_1 \right|^2 + 1}
\]

Notice, for forward scattering, \( \theta_{\text{com}} = 0 \), we have \( t = 0 \).

This is how \( t \) is defined.