Regge theory - Relativistic theory

- Allows getting a representation for $A(s,t)$ valid in all channel

Must make following assumptions for partial wave amplitudes.

1. $A_k(s)$ has only isolated singularities in complex $\ell$ plane (Regge poles)
2. $A_\ell(s)$ is holomorphic for $\text{Re} \, \ell \geq 1$
3. $A_\ell(s) \to 0$ as $|\ell| \to \infty$ for $\text{Re} \, \ell > 0$.

These are the postulate of maximal analyticity of the second kind.

Sommerfeld - Watson transformation —

recall from M.R. Scattering theory,

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell(s) P_\ell(z) = \frac{1}{2i} \oint_C \frac{(2\ell+1)A_\ell(s)P_\ell(-z)}{\sin \pi \ell} \, d\ell$$

Then provided assumptions 1, 2, 3 hold, can deform contour:

$$= \frac{1}{2i} \int_{-\frac{\pi}{2}-i\infty}^{-\frac{\pi}{2}+i\infty} \frac{(2\ell+1)A_\ell(s)P_\ell(-z)}{\sin(\pi \ell)} \, d\ell + 2\pi \sum_{i} \text{Res} \, \frac{A_\ell(s)P_\ell(-z)}{\sin(\pi \ell)} .$$

But in relativistic scattering theory do we know that $A_\ell(s) \to 0$ sufficiently quickly as $|\ell| \to \infty$ for $\text{Re} \, \ell > 0$ ?

Fortunately, we have the Froissart-Gribov representation that tells us something about large $\ell$ behavior.
\[ A_k(s) = \text{pole terms} + \frac{1}{\pi i} \int_{\gamma_{\text{out}}} dz' D_k(s, z') Q_k(z') + \frac{1}{\pi i} \int_{\gamma_{\text{in}}} dz' D_k(s, u(z')) Q_k(z') \]

In the second term, change var. \( z' \rightarrow -z' \), cf.

and use \( Q_k(-z) = -e^{-i\pi l} Q_k(z) \)\n
\[ Q_k(-z) = (-1)^{l+1} Q_k(z) \]

integer \( l \).

\[ A_k(s) = \text{pole terms} + \frac{1}{\pi i} \int_{\gamma_{\text{out}}} dz' D_k(s, z') Q_k(z') - e^{-i\pi l} \frac{\pi}{\pi} \int_{\gamma_{\text{in}}} dz' D_k(s, u(-z')) Q_k(z') \]

Now, as \( l \to \) complex infinity,

\[ Q_k \to \frac{1}{\pi i} e^{-\pi i (l+\frac{1}{2})} \text{sin}(\pi (z^2-1)^{1/2}) \quad \text{for} \quad \text{Re} \ l > \frac{1}{2} \]

but \( e^{-i\pi l} = e^{-i\pi (\text{Re} l + i \text{Im} l)} \)

\[ = e^{-i\pi \text{Re} l} e^{-i\pi \text{Im} l} \quad \text{blows up as} \quad \text{Im} l \to \infty . \]

As a result, condition 3 does not hold, because of the \((-1)^{l+1}\)

\[ \Rightarrow \text{cannot deform contour} \]

**Resolution:**

Just remove the \( e^{-i\pi l} \) factor and replace it with \( +1 \) or \(-1\)

For \( l = \) even integer, no difference \( \Rightarrow +1 \)

For \( l = \) odd integer, \( e^{-i\pi l} = -1 \) \( \Rightarrow -1 \)

So define two new partial wave amplitudes with definite signature \( \mathcal{S} = \pm 1 \)

\[ A^\mathcal{S}_k(s) = \text{pole terms} + \frac{1}{\pi i} \int_{\gamma_{\text{out}}} dz' D_k(s, z') Q_k(z') + \frac{\mathcal{S}}{\pi i} \int_{\gamma_{\text{in}}} dz' D_k(s, u(-z')) Q_k(z') \]

Then \( A^+_k(s) = A_k(s) \) for \( l \) even

\[ A^-_k(s) = A_k(s) \] for \( l \) odd.

This is how we continue the partial wave amplitudes into the complex \( s \)-plane.

By using the Fronsdorf-Gribov projection formula, and introducing two new scattering amplitudes, each with a definite signature.
\[ A_x(s) = \frac{1}{2} \sum_{l} (1 + \mathcal{S} e^{-i\pi L}) A_x^l(s) \]
\[ = \frac{1}{2} (A_x^+(s) + A_x^-(s)) + \frac{1}{2} e^{-i\pi L} (A_x^+(s) - A_x^-(s)) \]

\[ \text{this gives the pole term} \quad \text{and right-hand cuts} \]
\[ \text{this gives the left-hand cut} \quad \text{with the correct } \lambda \text{-dependence}. \]

Then, define the definite signature scattering amplitude

\[ A^\delta(s, t) = \sum_{l=0}^{\infty} (2l+1) A_x^{(l)}(s) P_l(z) \]
so that \( A_x^{(l)}(s) = \frac{1}{2} \int_{-1}^{1} dz A_x^\delta(s, z) P_l(z) \)

Now to construct the true full scattering amplitude, must remove \( \text{wrong-signatured terms} \) and replace with the right ones.

\[ A^+(s, z) \sim A_0^+(s) P_0(z) + A_1^+(s) P_2(z) + A_2^+(s) P_4(z) + \ldots \]
\[ A^-(s, z) \sim A_0^-(s) P_0(z) + A_1^-(s) P_2(z) - A_2^-(s) P_4(z) + \ldots \]

\( z \) stands in place of \( t \) using \( P_1(-z) = (-1)^{l} P_1(z) \).

\[ A^+(s, -z) \sim A_0^+(s) P_0(z) - A_1^+(s) P_2(z) + A_2^+(s) P_4(z) + \ldots \]
\[ A^-(s, -z) \sim A_0^-(s) P_0(z) - A_1^-(s) P_2(z) + A_2^-(s) P_4(z) + \ldots \]

Thus the full scattering amplitude is:

\[ A(s, t) = \frac{1}{2} \left[ (A^+(s, z) + A^+(s, -z)) + (A^-(s, z) - A^-(s, -z)) \right] = \]
\[ \text{gives the right-signatured even-} \lambda \text{-terms} \quad \text{gives the right-signatured odd-} \lambda \text{-terms}. \]

Since \( A_x^{(l)}(s) \), and hence \( A^\delta(s, t) \) have good \( \lambda \rightarrow \infty \) behavior,
so does our definition of \( A(s, t) \). Perform Sommerfeld-Watson transform on this.
\[ \frac{1}{2} \sum_{\lambda} A^\lambda(s,t) + J^\lambda(s,-t) \]

Sommerfeld-Watson of \( A^\lambda(s,t) \) [s-channel]

\[ A^\lambda(s,t) = \sum_{l=0}^{\infty} (2l+1) A^\lambda_l(s) P_l(z) \]

\[ \frac{1}{2i} \oint_C \frac{d\lambda}{\sin \pi \lambda} (2\lambda+1) A^\lambda(s) P_\lambda(-z) \]

\[ = \frac{1}{2i} \int_{-\infty}^{\infty} d\lambda \frac{(2\lambda+1) A^\lambda(s) P_\lambda(-z)}{\sin(\pi \lambda)} + 2\pi \sum_{\epsilon} \frac{2\alpha_\epsilon^2(s) + 1}{\sin(\pi \alpha_\epsilon^2(s))} \frac{\beta(s) P_{\alpha_\epsilon^2(s)}(-z)}{\sin(\pi \alpha_\epsilon^2(s))} \]

Sum over definite signatureed Regge poles.

Now, all three assumptions hold \( \Rightarrow \) deform contour

\[ = \frac{1}{2i} \int_{-\infty}^{\infty} d\lambda \frac{(2\lambda+1) A^\lambda(s) P_\lambda(-z)}{\sin(\pi \lambda)} + 2\pi \sum_{\epsilon} \frac{2\alpha_\epsilon^2(s) + 1}{\sin(\pi \alpha_\epsilon^2(s))} \frac{\beta(s) P_{\alpha_\epsilon^2(s)}(-z)}{\sin(\pi \alpha_\epsilon^2(s))} \]

Then using \( A(s,t) = \frac{1}{2} \sum_{\lambda} [A^\lambda(s,z) + J^\lambda A^\lambda(s,-z)] \),

and using \( P_\lambda(z) = (-1)^l P_\lambda(-z) \equiv e^{-i\pi \lambda} P_\lambda(-z) \) on second term

\[ A(s,t) = \frac{1}{2} \sum_{\lambda} \frac{1}{2i} \int_{-\infty}^{\infty} d\lambda \frac{(2\lambda+1) P_\lambda(-z)}{\sin(\pi \lambda)} (1 + J^\lambda e^{-i\pi \lambda}) A^\lambda(s) \leftarrow \text{Background integral} \]

\[ + \frac{1}{2} \sum_{\lambda} 2\pi \sum_{\epsilon} \frac{2\alpha_\epsilon^2(s) + 1}{\sin(\pi \alpha_\epsilon^2(s))} \frac{\beta(s) P_{\alpha_\epsilon^2(s)}(-z)}{\sin(\pi \alpha_\epsilon^2(s))} \leftarrow \text{Regge pole term} \]

Signature factor

In the large \( t \) limit, the Regge pole part gives the dominant contribution:

\( \alpha \) fixed.

\[ P_\alpha(-z) \xrightarrow{t \to \infty} (-z)^\alpha \] [Regge limit] \( z = \frac{t-u}{4|p_1|^2} + O(\frac{1}{s^2}) \)

keeping the leading pole (\( \Re \alpha \) greatest)

\[ \Rightarrow A(s,t) \xrightarrow{t \to \infty} -\beta(s) \left( 1 + J^\lambda e^{-i\pi \alpha(s)} \right) \left( \frac{t-u}{4|p_1|^2} \right)^\alpha(s) \]

This limit is strictly academic, but by doing the same thing in the crossed channel, we can learn about the large \( s \) behaviour of the direct-channel scattering amplitude.
Sommerfeld-Watson transform [t-channel]

\[ A^\gamma(s, t) = \sum_{l=0}^{\infty} (2l+1) A_l^\gamma(t) P_l(z_t) \]

\[ z_t = \cos \theta_t = \frac{s-u}{|p_1|^2 + p_2^2} + \mathcal{O}\left(\frac{\Delta m^2}{s}\right) \]

\[ = \frac{1}{2i} \oint_C dl \frac{(2l+1) A_l^\gamma(t) P_l(-z_t)}{\sin \pi l} \]

Deform contour, exposing Regge poles

\[ = \frac{1}{2i} \int_{-\infty}^{\infty} dl \frac{(2l+1) A_l^\gamma(t) P_l(-z_t)}{\sin \pi l} + 2\pi \sum_i (2\alpha_i^\gamma(t) + 1) \frac{\beta(t) P_{\alpha_i^\gamma}(t)}{\sin(\pi \alpha_i^\gamma(t))} \]

Then, use

\[ A(s, t) = \frac{1}{2} \sum_j \left[ A^\gamma(z_j, t) + \int A^\gamma(\zeta, t) \right] \]

and

\[ P_l(-z) = (-1)^l P_l(z) = e^{-i\pi l} \]

\[ A(s, t) = \frac{1}{2} \sum_j \frac{1}{2i} \int_{-\infty}^{\infty} dl \frac{(e^{-i\pi l} + \gamma)}{\sin(\pi l)} A_l^\gamma(t) P_l(z) \]

Physical consequence:

\[ z_t = \frac{s-u/4}{|p_1|^2 + p_2^2} + \mathcal{O}\left(\frac{\Delta m^2}{s}\right) \]

Usually, define \( u = \frac{s-u}{2} \)

In the large \( s \), fixed \( t \), limit (Regge limit)

\[ z_t \rightarrow \frac{s}{s_0} \]

and

\[ p_\alpha(t) \xrightarrow{t \rightarrow \infty} \Re \alpha \]

In the extreme large \( s \) limit, leading pole (largest \( \Re \alpha \)) dominates.

Diagrammatically:

Diagram with vertices and lines connecting them, indicating the flow of \( A(s, t) \) as \( s \to \infty \).