Narrow Width Approximation (NWA)  \[ \text{[S-channel factorization]} \]

If scattering proceeds via a resonance (of mass \(m_R\) and width \(\Gamma\)), one can "factorize" the differential cross section into a differential production cross section \(d\sigma_{\text{prod}}\) and differential decay rate \(d\Gamma_{\text{decay}}\).

Result:
\[
d\sigma_{\text{NWA}} = d\sigma_{\text{prod}} \times \frac{d\Gamma_{\text{decay}}}{\Gamma_{\text{decay}}} + \frac{\Gamma_{\text{decay}}}{m_R} \frac{d^2E}{d^2Q}.
\]

WARNING! There are many sources of error to this approximation; not just \(\Gamma/m_R\).

FORMALISM (illustrates use of recurrence relation for many-body phase space).

Consider: \(AB \rightarrow (12)3\)

Differential cross section:
\[
d\sigma = \frac{1}{F_{\text{max}}} | M_{AB \rightarrow 123} |^2 d(E_{P_2}; Q; p_1, p_2, p_3)
\]

Since particles 1 \& 2 are produced in resonance, use phase space recurrence relations: group particles 1 \& 2 together and associate fictitious particle with resonance.

Use "unitarity" to factorize scattering amplitude at resonance pole.
\[
| M_{AB \rightarrow 123} |^2 = "\" | M_{AB \rightarrow k,3} |^2 \frac{1}{k^2 - m_R^2 + i\Gamma/2} \frac{1}{M_{k=1,2}} |^2
\]

*Unitarity allows this for strictly stable particles. Thus, an approximation is already made at this point, and this approximation violates unitarity.

This is directly related to the fact that scattering amplitudes are defined for strictly stable external particles only.

Now use recurrence relation of phase space, grouping particles 1 \& 2:
\[
d(E_{P_2}; Q; p_1, p_2, p_3) = \sum_{M_{k=1,2}}^{|M_{k=1,2}|^2} \int_{M_{k=1,2}}^{|M_{k=1,2}|^2} d(E_{P_2}; Q; k, p_3) \int_{M_{k=1,2}}^{|M_{k=1,2}|^2} d(E_{P_2}; Q; k, p_2)\]
So that differential cross section reads:

\[ d\sigma = \frac{1}{4\pi} \int_{M_{\text{min}}}^{M_{\text{max}}} \frac{dM^2}{2\pi} |M_{AB} - k_3|^2 \frac{1}{k^2 - m_\omega^2 + i\Gamma m_\omega} |M_{k \to \gamma}|^2 \left( \frac{1}{M^2 - m_\omega^2 + i\Gamma m_\omega} \right)^2 \left( \text{Jacobian factor in phase space} \right) \times d(LIPS_2; \Omega; k, p_3) \]

Assume \( f(q, p_2, p_3) = |M_{AB} - k_3, M_{k \to \gamma}|^2 \times (\text{Jacobian factor in phase space}) \) is approximately constant (slowly varying function of \( k^2 \)) over the narrow width of Breit-Wigner resonance.

Use \( \delta_+(k^2 - M^2) \) in \( d(LIPS_2; \Omega, k, p_3) \) to replace \( k^2 - M^2 \) in resonance propagator.

\[ d\sigma = \frac{1}{4\pi} \int_{M_{\text{min}}}^{M_{\text{max}}} \frac{dM^2}{2\pi} |M_{AB} - k_3|^2 \left( \frac{1}{M^2 - m_\omega^2 + i\Gamma m_\omega} \right)^2 \left( \text{Jacobian factor in phase space} \right) \times |M_{k \to \gamma}|^2 \left( \text{Jacobian factor in phase space} \right) \times d(LIPS_2; k, p_3, p_2). \]

Extend integration range: \( M_{\text{min}} \to -\infty ; M_{\text{max}} \to +\infty \)

assuming that unphysical region will have negligible impact due to Breit-Wigner tail.

Then, observe:

\[ I = \int_{-\infty}^{\infty} \frac{dM^2}{2\pi} \left| \frac{1}{M^2 - m_\omega^2 + i\Gamma m_\omega} \right|^2 f(M^2 \approx m_\omega^2) \]

\[ = \int_{-\infty}^{\infty} \frac{dM^2}{2\pi} \frac{1}{(M^2 - m_\omega^2)^2 + m_\omega^2 \Gamma^2} f(M^2 \approx m_\omega^2) \]

change var: \( M^2 \to m_\omega^2 \chi \)

\[ = \int_{-\infty}^{\infty} \frac{dx}{2\pi} m_\omega^2 \frac{1}{m_\omega^2 (x - 1)^2 + m_\omega^2 \Gamma^2} f(m_\omega^2) \]

\[ = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{m_\omega^2}{\Gamma^2} \frac{1}{(\frac{m_\omega^2}{\Gamma})^2 (x - 1)^2 + 1} f(m_\omega^2) \]

\[ = \frac{1}{\Gamma^2} \frac{1}{2(\frac{m_\omega}{\Gamma})} f(m_\omega^2) = \frac{1}{2m_\omega \Gamma} f(m_\omega^2) \]

(Note: if next term in Taylor expansion of \( f(M^2) \) is retained, one would find the \( O(1/m_\omega) \) dependence of error.)
Thus, the differential cross section reads:

\[
\frac{d\sigma}{d\Omega} = \frac{1}{\text{Flux}} \frac{1}{2m_w} \left| M_{\theta \to \gamma_3} \right|^2 d\left(\hat{Q}_3, k_3, p_3\right) \cdot \left| M_{\gamma_3 \to \gamma_3} \right|^2 d\left(\hat{Q}_3, k_3, p_3, p_2\right)
\]

where \[
\frac{d\Gamma}{\Gamma} = \frac{1}{2m_w} \left| M_{\gamma_3 \to \gamma_3} \right|^2 d\left(\hat{Q}_3, k_3, p_3, p_2\right).
\]

Comments:

Strictly speaking, there is no such thing as an "on-shell" particle that can decay. We can never have \( k^2 = m^2 \) pole -

\[\text{real-valued}\quad \text{complex-valued}\]

Such unusual language belongs in the realm of approximations.

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- Need to study case when resonance particle has spin.
  (Use old-fashioned \(\pi\) to analyze?)

\[D^{\pi^0} = \sum_{n=1}^{\infty} \epsilon^0_n F^0_n\]

Probable need \(\epsilon_n^\pi(\pi)\) & \(\epsilon_n^\pi(\pi)\) conformal quantized formulation

of gauge theory.

- Make analogy with Z-the experiment:

\[\]
Error associated with NWA:  (Berdin, Kaiser, Rainwater)

\[ \Gamma \ll m_\ast \quad \text{Approx \#2} \]

\[ m_1, m_2 \ll m_\ast \quad \text{Approx \#3} \]

\[ \sqrt{S} \gg m_\ast \quad \text{set by } \sqrt{S} \text{ and limited by } m_2 \text{ & } m_3 \]

no interference, no competing channels (e.g. $e^+e^-\rightarrow j^+j^-$, $Z\rightarrow j^+j^-$)

separable pp. \quad \text{Approx \#1}

Unfortunately, this means an accurate reconstruction of the Lagrangian requires a careful evaluation of the cross section with the entire matrix element – leading to a justification of the use of (black-box) MC simulators.