Examples of singularities. (In complex variable)

1. \[ F(s) = \int_a^b \frac{dx}{x-s} = \ln \left( \frac{b-s}{a-s} \right) \]

   \[ \frac{1}{x-s} \]

   \[ a \quad b \]

   if singularity reaches a or b,
   
   \[ F(s) \] becomes singular \quad Endpt Singularity

2. \[ F(s) = \int_0^1 \frac{dx}{(x-s)(x-\alpha)} = \frac{1}{s-\alpha} \ln \left( \frac{\alpha(1-s)}{(1-\alpha)s} \right) \]

   \[ \frac{1}{(x-s)(x-\alpha)} \]

   \[ 0 \quad 1 \quad \alpha \]

   When singularity at
   \[ x=s \] reaches the
   singularity at \[ x=\alpha, \]

   integral becomes
   singular.

   Pinch-singularity

What if \[ x=s \] singularity approached \( x=\alpha \) singularity from the other side?

\[ \ln (1) = 0 \quad \text{or} \quad = \pm 2\pi i ? \]

Case at LEFT

\[ \frac{\alpha}{\alpha-\alpha} \]

\[ \text{converge singularity} \]

leaving finite part

[PRINCIPAL SENSE]

Case ABOVE

\[ \text{no cancellation} \]

of singularity.

3. \[ F(s) = \int_2^3 \frac{dx}{x^5+1} = \frac{1}{5} \ln \left( \frac{3s+1}{2s+1} \right) \]

   \[ s=-\frac{1}{2} \quad \text{or} \quad -\frac{1}{3} \Rightarrow \text{End-pt singularity} \quad s=0 \]
Generally, singularities are usually branch points.

Determining discontinuity across branch points:

\[ F(s) = \int_A^B dx \, f(s, x) \]  
Suppose \( f(s, x) \) has one pole 
at \( x = x_{pole}(s) \)

\[ A \rightarrow x_{pole}(s) \rightarrow B \]  
\( \Rightarrow F(s) \) has two end-point singularities.

Take \( s \) so that pole is very close to end-point \( A \).
Allow \( s \) to vary, so that it makes one revolution about \( A \) \( \Rightarrow \) must deform contour.

\[ \text{Heuristically, discontinuity of } F(s) \text{ associated with singularity at } A \]
is just \( 2\pi i \cdot \text{Residue of pole evaluated at } s \text{ such that pole is at } x = A \).

Same discontinuity is picked up each time pole encircles \( A \).
\( \Rightarrow \) singularity is logarithmic.
More Examples

\#1  \[ F(s) = \int_{-1}^{+1} \frac{dx}{x^2 + s} \]

End-point singularities: \( s = \pm 1 \)

\[ (-1)^2 + s = 0 \Rightarrow s = -1 \]

\[ s = \sqrt{\text{Re}(s)^2 + \text{Im}(s)^2} \]

Write \( s = r e^{i\theta} \)

\[ r = \sqrt{\text{Re}(s)^2 + \text{Im}(s)^2} \]

\[ \sqrt{1} = \sqrt{1} \]

\[ s = 0 \]

\[ \text{pinch singularity at } s = 0 \]

\[ r \sqrt{s} = \sqrt{r \cos \frac{\theta + \pi}{2}} \]

\[ \text{Im } s = \sqrt{r \sin \frac{\theta + \pi}{2}} \]

\[ F(s) = \int_{-1}^{+1} \frac{dx}{x^2 + s} = \frac{\pi}{\sqrt{s}} \tan^{-1} \left( \frac{1}{\sqrt{s}} \right) \]

\[ = \frac{\pi}{\sqrt{s}} \left[ \ln \left( 1 - \frac{i}{\sqrt{s}} \right) - \ln \left( 1 + \frac{i}{\sqrt{s}} \right) \right] \]

\[ = \frac{\pi}{\sqrt{s}} \ln \left( \frac{1 - i/\sqrt{s}}{1 + i/\sqrt{s}} \right) \]

\[ x^2 = -s \]

\[ x = \pm i\sqrt{s} \Rightarrow \text{if } s < 0 \quad x = \pm 1 \]

End-point singularities:

When \( s^2 = -s \Rightarrow s = -1 \)

\[ \text{Log } s = \pi i \]

Pinch singularities: clearly when \( s = 0 \).

\[ F(s) \uparrow \]

\[ G(s, x) = x^2 + s \]

Solve \( x^2 = -s \quad s = 0 \).

And \( \frac{\partial C}{\partial y} = 2x = 0 \Rightarrow x = 0 \)
\[ F(s) = \int_0^1 \frac{dx}{(x-z)(x-s)} = \frac{1}{s-2} \ln \left( \frac{2(s-1)}{s} \right) \]

Search for end-point singularities:

At \( x = 0 \):
\[ (0-z)(0-s) = 0 \quad \Rightarrow \quad s = 0 \]
End point singularity

At \( x = \infty \):
\[ \frac{dx}{(1-z)(1-s)} \quad \Rightarrow \quad s = 1 \]

Search for pinch singularities:

\[ f(x,s) = (x-z)(x-s) = 0 \]

\[ \frac{\partial}{\partial x} f = \frac{2}{\partial x} (x-z)(x-s) = 0 \]

\[ -2 - 5 + 2x = 0 \]

\[ x = 1 + \frac{s}{2} \]

\[ (1 + \frac{s}{2} - z)(1 + \frac{s}{2} - s) = 0 \]

\[ (-1 + \frac{s}{2})(1 - \frac{s}{2}) = 0 \]

\[ (1 - \frac{s}{2})^2 = 0 \]

\[ s = 2 \quad \text{(double root)} \]

Pinch singularity
\# 3 \quad F(s) = \int_0^1 \frac{dx}{1 - x^2 s} \quad \text{for } \ s \neq 0 \quad \text{and } \ x \neq \pm \frac{1}{\sqrt{s}}

\text{Search for end-point singularities.}

Set \ x = 0: \quad \text{Denom} = 1 - 0^2 s = 0
\quad 1 = 0 \quad \text{(no end-point singularity associated with } \ x = 0)\n
Set \ x = 1: \quad \text{Denom} = 1 - 1^2 s = 0
\quad s = 1 \quad \text{(end-point singularity at } s = 1)\n
\text{Search for pinch singularities (double root of } G(s, x) = 0)\n
1 \quad G(s, x) = 1 - x^2 s = 0 \quad \Rightarrow \quad x = \frac{1}{\sqrt{s}}

2 \quad \frac{\partial G}{\partial x} = -2x s = 0 \quad \Rightarrow \quad \text{Either } x = 0 \text{ or } s = 0

Notice however, if \ s = 0, \text{ location of pole in } x\text{-plane goes to infinity. In certain cases, can drag contour to infinity with it,}

\text{result satisfies 1), so no pinch singularities.}

\quad F(s) = \int_0^1 \frac{dx}{1 - x^2 s} = \frac{1}{1^2} \tanh^{-1} \sqrt{s} = \frac{1}{2s} \ln \left( \frac{1 + \sqrt{s}}{1 - \sqrt{s}} \right)
More than one external variable

Example. \( F(s,t) = \int dx \, f(s, t, x) \)

Locations of the singularities of \( f(s, t, x) \) in complex \( x \)-plane are functions of \( s \) & \( t \): pole \((s, t)\). Still two sources of singularities—end-point and pinch.

Location of pole of \( f(s, t, x) \) in complex \( x \)-plane, \( x_{\text{pole}}(s, t) \)

may be a root of an auxiliary function \( G(s, t; x) \), so that

\( G(s, t; x_{\text{pole}}(s, t)) = 0 \) is satisfied.

Two poles:

\[
\begin{align*}
G(s, t; x^{(1)}_{\text{pole}}(s, t)) &= 0 \\
G(s, t; x^{(2)}_{\text{pole}}(s, t)) &= 0
\end{align*}
\]

In order for pinch to occur, poles must coincide \( \Rightarrow x_{\text{pole}}(s, t) \)

must be a double root of \( G(s, t; x) \)

So \( G(s, t; x_{\text{pole}}(s, t)) = 0 \)

AND

\[
\frac{\partial G(s, t)}{\partial x} \bigg|_{x_{\text{pole}}} = 0.
\]

If pinch occurs due to three singularities that coincide, then we have, in addition,

\[
\frac{\partial G(s, t)}{\partial x} \bigg|_{x_{\text{pole}}} = 0 \text{ at this point } (x_{\text{pinch}}).
\]

The location of poles are a function of 2 complex numbers, \( s \) & \( t \). But, the requirement

\( G(s, t; x_{\text{pole}}^{(2)}(s, t)) = 0 \)

puts 2 (one on real part of \( G \), the other on imaginary part) constraints.

\( \Rightarrow \) pole is a 2-dimensional surface in a 2x2=4 dimensional space.
More than one integral (integration) variable

With many Feynman parameters, the integration is a region, subject to \( x_1 + x_2 + \ldots + x_N = 1 \)

\[ \int dx_1 \ldots dx_N \ f(s,t; \ldots; x_1 \ldots x_N) \text{number of propagators} \]

There may be many external lines with momenta \( p_1^\mu, p_2^\mu, \ldots, p_N^\mu \)

- There are \( 4 \times N \) components
- \( \Rightarrow \) but 4 lost due to 1 Energy & 3 momentum conservation relations (covariant symmetry)
- \( \Rightarrow N \) on-shell relations \( p_i^2 = m_i^2 \)
- \( \Rightarrow \) Amplitude invariant under rotations (3) & boosts (3)

Hence scattering amplitude characterized by \( 4N - 4 - N - 6 = 3N - 10 \)

2\( \to \) 2 scattering: \( N = 4 \Rightarrow 3(4) - 10 = 2 \) invariants,

\[ \Rightarrow \text{e.g. Mandelstam: } s, t, u \]

(are redundant: \( s + t + u = 2m^2 \))

2\( \to \) 3 scattering: \( N = 5 \Rightarrow 3(5) - 10 = 5 \) invariants,

\[ \Rightarrow \text{e.g. } \begin{align*}
(p_2 + p_3)^2 &= Q^2 \\
(p_3 + p_4)^2 &= m_3^2 \\
(p_2 - p_3)^2 &= m_2^2 \\
(p_4 + p_5)^2 &= m_4^2 \\
(p_1 - p_5)^2 &= \text{bound state}.
\end{align*} \]

Surface of singularity is \( G(s,t; \ldots; x_1, x_2, \ldots, x_N) = 0 \)

In the \( 2N \) dimensional space, the singularity surface is \( 2N - 2 \) dimensional.

The integration contour is an \( N \) dimensional surface, \( \mathbb{N} \).
More Than One Integration Variable

There are four sources of singularities in complex \((s, t, \ldots)\) plane after integration.

1. Two singularity surfaces \(S_1\) & \(S_2\) in complex \((x_1, \ldots, x_N)\) plane advance on integration contour, pinching it at the same point.

\[
G(s, t, \ldots; \overline{x}_{\text{pole}}(s, t, \ldots)) = G(s, t, \ldots; \overline{x}_{\text{pole}}^{(2)}(s, t, \ldots)) = 0
\]

[Topology/ Homology]

\[
\left\{ \begin{array}{l}
\alpha_1 G(x_{\text{pole}}) + \alpha_2 G(x_{\text{pole}}^{(2)}) = 0 \\
\alpha_1 \frac{\partial G}{\partial x_i} \bigg|_{x_{\text{pole}}} + \alpha_2 \frac{\partial G}{\partial x_i} \bigg|_{x_{\text{pole}}^{(2)}} = 0 \quad (\text{for all } i = 1, \ldots, N)
\end{array} \right.
\]

for some \(\alpha_1\) & \(\alpha_2\).

[Coincidence of normals condition]

2. Two different parts of same singularity surface can trap integration contour:

\[
G(s, t, \ldots; \overline{x}_{\text{pole}}(s, t, \ldots)) = 0
\]

\[
\frac{\partial G}{\partial x_i} \bigg|_{x_{\text{pole}}} = 0 \quad \text{[double root condition]}
\]

3. Three (or more) singular surfaces may pinch contour (generalization of 2)

\[
G(s, \ldots; \overline{x}_{\text{pole}}^{(r)}) = G(s, \ldots; \overline{x}_{\text{pole}}^{(r)}) = \ldots = G(s, \ldots; \overline{x}_{\text{pole}}^{(r)}) = 0
\]

\[
\alpha_1 \frac{\partial G}{\partial x_i} \bigg|_{x_{\text{pole}}^{(1)}} + \ldots + \alpha_r \frac{\partial G}{\partial x_i} \bigg|_{x_{\text{pole}}^{(r)}} = 0 \quad (\text{for all } i = 1, \ldots, N)
\]

for some \(\alpha_1, \ldots, \alpha_r\).

4. Singularity surface hits boundary of integration (analogy of end-point singularity). If boundary surface is \(B_i(s, t, \ldots; x_i, x_N) = 0\) \((N-1\) dimension).

In the case of Feynman integral, \(B_i = x_i = 0\)

[Note: \(x_i = 1\) wiht a boundary, because of delta function, \(x_i = 0\).]

\[
\alpha_1 B_1 + \ldots + \alpha_N B_N = 0 \quad \alpha_1 \frac{\partial B_i}{\partial x_i} + \ldots + \alpha_N \frac{\partial B_i}{\partial x_N} = 0
\]
Summary: The equations are
\[ \bar{\alpha}_1 B_1 + \ldots + \bar{\alpha}_n B_n = 0 \]
"End point (Boundary) singularity"
(road deform contour)

\[ \alpha_1 \frac{\partial G_1}{\partial \xi} + \ldots + \alpha_N \frac{\partial G_N}{\partial \xi} + \bar{\alpha}_1 \frac{\partial B_1}{\partial \xi} + \ldots + \bar{\alpha}_n \frac{\partial B_n}{\partial \xi} = 0 \]
"Pinch Singularity"

\[ G = 0 \] singularity surface equation
\[ B = 0 \] Boundary equation (for region of integration).

Definitions

Leading Singularity: Singularity arising in the amplitude when all parameters (\( \alpha_i \) or \( \xi_i \)) are non-zero.

Subleading Singularity: Singularity arising when at least one parameter (\( \alpha_i \) or \( \xi_i \)) is zero. The propagator associated with vanishing \( \alpha \) or \( \xi \) is "short-circuited." Sub-leading singularity is the leading singularity of associated diagram with the short-circuited propagator.

\( \chi_4 = 0 \).

Normal Threshold: Singularity expected from unitarity.

Anomalous Threshold: Unexpected singularity. Not derived from unitarity considerations.
There are three useful representations of Scalar Feynman diagrams

1. (Diagram) \[ \int \frac{d^4 k_2}{(2\pi)^d} \cdot \frac{d^4 k_1}{(2\pi)^d} \frac{1}{(q_1^2 - m_1^2) \cdots (q_N^2 - m_N^2)} \]

Set \( G_1 = q_1^2 - m_1^2, \quad G_2 = q_2^2 - m_2^2, \ldots \)

\[ \alpha_1 G_1(k_1, \ldots, k_N) + \alpha_2 G_2(k_1, \ldots, k_N) + \ldots = 0 \quad \Rightarrow \quad \text{either} \quad q_1^2 = m_1^2 \quad \text{or} \quad \alpha_1 = 0 \]

\[ \alpha_1 \frac{\partial G_1}{\partial k_1} + \alpha_2 \frac{\partial G_2}{\partial k_2} + \ldots \]

\[ \Rightarrow \quad \alpha_1 \frac{\partial}{\partial k_i} (\frac{1}{q_i^2 - m_i^2}) + \alpha_2 \frac{\partial}{\partial k_i} (\frac{1}{q_i^2 - m_i^2}) + \ldots = 0 \]

But since \( q_i^2 \) is a linear function of the loop momenta, \( k_i^\mu \),

\[ \Rightarrow \quad \alpha_1 2 q_{1\mu} \frac{\partial q_1^\nu}{\partial k_1^\mu} + \alpha_2 2 q_{2\mu} \frac{\partial q_2^\nu}{\partial k_2^\mu} + \ldots = 0 \]

\[ \Rightarrow \quad 2 \left( \alpha_1 q_1^\mu + \alpha_2 q_2^\mu + \ldots \right) = 0 \]

\[ \sum_{i=1}^{N} \alpha_i q_i^\mu = 0 \]

Landeau Equations for first representation

2. (Diagram)

\[ \int_0^\frac{1}{\pi} dx_1 \ldots dx_N \int \frac{d^4 k_1}{(2\pi)^d} \cdot \frac{d^4 k_2}{(2\pi)^d} \frac{(N-1)!}{(2\pi)^d} \delta(1-x_1 - \ldots - x_N) \]

Only one \( G \):

\[ G = \left[ x_1 (q_1^2 - m_1^2) + \ldots + x_N (q_N^2 - m_N^2) \right] \quad B_i \equiv \delta i = 0 \]

\[ G(k_1, \ldots, k_N, x_1, \ldots, x_N) = 0 \quad \text{Either} \quad \alpha = 0 \quad \text{or} \quad G = 0 \]

\[ \frac{\partial G}{\partial k_1^\mu} + \alpha_1 \frac{\partial B_1}{\partial k_1^\mu} + \ldots + \alpha_N \frac{\partial B_N}{\partial k_1^\mu} = 0 \quad \Rightarrow \quad \alpha \frac{\partial G}{\partial k_1^\mu} = 0 \]

\[ \frac{\partial G}{\partial x_1^\mu} + \alpha_1 \frac{\partial B_1}{\partial x_1^\mu} + \ldots + \alpha_N \frac{\partial B_N}{\partial x_1^\mu} = 0 \]

\[ \alpha \frac{\partial G}{\partial x_i^\mu} = 0 \quad \text{Same as representation 1} \]
\[ \sum_{i=1}^{N} \alpha_i \frac{\partial G}{\partial x_i} = G(x_i \to \alpha_i) \]

Hence, these two equations are equivalent.
Landau Conditions

\[ \Delta = \frac{1}{2} \sum_{i,j} X_i X_j Y_{ij} \quad \text{Denominator function.} \]

In order for singularity to occur need:

\[ \Delta = 0 \quad (x) \quad \text{AND one of} \]

1. all of the \( X \)'s vanish \( X_i = 0 \)
   (red-point singularity) - singularity on all sheets.

2. \( \frac{\partial \Delta}{\partial X_i} = 0 \) for all \( X_i \)
   (pinch singularity) - gives branch point (on one sheet) or pole

The solutions to \( 1 \times 2 \) all into two classes:

Class [A]: Threshold (anomalous and physical)

The conditions give \( X_i \) as pinch of internal masses & momenta:

\[ X_1 \equiv x_3 (s_3j, m_3) \]
\[ X_2 \equiv x_2 (s_2j, m_2) \]

And only at isolated values for all invariants are Landau conditions satisfied.

Class [B]: IR divergence - soft and collinear

Solutions at isolated values for a subset of invariants, and arbitrary values for all others.

At these isolated values for the subset \( m \), \( (e.g. p_1^2 = m_1^2 = m_2^2 = 0 \; \text{etc}...) \)
we have IR divergence.
Example:

\[
\begin{array}{cc}
\begin{array}{c}
S \quad \frac{p}{k} \quad \frac{p}{p-k}
\end{array}
\end{array}
\quad = -i \sum (p^2) \sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}
\]

Consider the \( k^0 \) integral. In the complex \( k^0 \)-plane, there are the following poles, from each propagator:

1. \( k^2 - m^2 + i\epsilon = 0 \Rightarrow k^0 = \pm \sqrt{k^2 + m^2} \pm i\epsilon \) [shown as 'x' in plot]

2. \( (p-k)^2 - m^2 + i\epsilon = 0 \Rightarrow k^0 = \pm \sqrt{4(p^0)^2 - 4((p-k)^2 - m^2 + i\epsilon)} \)

\[
= p^0 \pm \sqrt{(p-k)^2 + m^2} \pm i\epsilon \quad [\text{shown as 'x' in plot}]
\]

The poles are (generically) shown above. Can Wick rotate to integrate along imaginary \( k^0 \)-axis.

But, notice: as \( p^0 \) (external energy) is increased, two poles \( \otimes \) drift to the right. When it runs into the integration contour, the contour may be deformed (next page).

Once the \( +\sqrt{k^2+m^2} \) \& \( p^0 - \sqrt{(p-k)^2 + m^2} \) poles line up vertically, the \( \epsilon \to 0 \) limit will pinch the contour \( \Rightarrow \) integration ill-defined \( \Rightarrow \) singularity.
Solve for $p^0$ to find when pinch occurs,

$$\sqrt{E^2 + m^2} = p^0 - \sqrt{(p^0 - E)^2 + m^2}$$

$$\Rightarrow p^0 = \sqrt{E^2 + m^2} + \sqrt{(p^0 - E)^2 + m^2}$$

Boost to rest frame of particle, so that $p'^0 = (p^0, p^0 = 0)$.

Then singularity occurs at $p^0 = 2\sqrt{E^2 + m^2} \approx 2$ particle threshold.

"Threshold singularity" corresponds to branching point. Can calculate imaginary part:

- Rest to leave integration contour unrotated.
- Close contour in upper-half plane, picking up two poles at

$$k^0 = -\sqrt{E^2 + m^2} + i\epsilon$$

and

$$k^0 = p^0 - \sqrt{(E - p^0)^2 + m^2} + i\epsilon$$
Triangle Graph:

\[
\begin{align*}
q_1 &= p_1 + k \\
q_2 &= p_1 + p_2 + k \\
q_3 &= p_1 - q_1 = -k
\end{align*}
\]

\[
I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(q_1^2 - m_1^2 + i\epsilon)(q_2^2 - m_2^2 + i\epsilon)(q_3^2 - m_3^2 + i\epsilon)}
\]

\[
= \int dx_1 dx_2 dx_3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[x_1(q_1^2 - m_1^2) + x_2(q_2^2 - m_2^2) + x_3(q_3^2 - m_3^2)]}
\]

\[
[... ] = k^2 - 2k \left(-\gamma_1 - \gamma_2, -\gamma_3\right) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} p_1, p_2 \end{pmatrix} \left(\begin{array}{cc}
\gamma_1 + \gamma_2 & \gamma_2 \\
\gamma_2 & \gamma_2
\end{array}\right) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}
\]

\[A = 1\]

\[B = (-\gamma_1 - \gamma_2, -\gamma_3)\]

\[\Gamma = \begin{pmatrix} \gamma_1 + \gamma_2 & \gamma_2 \\
\gamma_2 & \gamma_2
\end{pmatrix}\]

Upon doing momentum integrals, we have:

\[
I \sim \int dx_1 ... dx_3 \times \left(1 - x_1 - x_2 - x_3\right) \left[p_1, p_2\right] \left[p_1, p_2\right] \frac{1}{[p_1, p_2 - \gamma_1, \gamma_2]}
\]

\[D = \begin{pmatrix} (\gamma_1 + \gamma_2)(1 - x_1 - x_2) & x_2(1 - x_1 - x_2) \\
x_2(1 - x_1 - x_2) & x_2(1 - x_2)
\end{pmatrix} = \begin{pmatrix} (\gamma_1 + \gamma_2)x_3 & x_2x_3 \\
x_2x_3 & x_2(1 - x_2)
\end{pmatrix}\]

\[\text{det}_D = \begin{pmatrix} p_1 & p_2 \\
x_2x_3 & x_2(1 - x_2)
\end{pmatrix} = x_1m_1^2 - x_2m_2^2 - x_3m_3^2 + i\epsilon\]
Subleading singularities

Take \( x_3 = 0 \implies x_4 + x_2 = 1 \) or \( x_2(x_3) = 1 - x_4 \)

\[
\text{denom} = (p_2 \ p_2) \begin{pmatrix} 0 & 0 \\ 0 & x_2(1-x_2) \end{pmatrix} (p_2) - x_2 m_2^2 - x_2 m_2^2 + i\epsilon
\]

\[
= x_2 x_1 p_2^2 - x_2 m_2^2 - x_2 m_2^2 + i\epsilon
\]

\[
\frac{\partial \text{denom}}{\partial x_1} = (1 - 2x_4)p_2^2 - m_2^2 + m_2^2 = 0 \implies x_1 = \frac{m_2^2 + m_2^2}{2p_2^2} + \frac{1}{2}
\]

\[
\frac{\partial \text{denom}}{\partial x_2} = (1 - 2x_2)p_2^2 + m_2^2 - m_2^2 = 0 \implies x_2 = \frac{m_2^2 - m_2^2}{2p_2^2} + \frac{1}{2}
\]

Note \( x_1 + x_2 = 1 \)

Plug solution back into denom and solve for \( p_2^2 \):

\[
\text{denom} = \left( \frac{m_2^2 - m_2^2}{2p_2^2} + \frac{1}{2} \right) \left( -m_2^2 + m_2^2 \right) p_2^2 - \left( \frac{m_2^2 + m_2^2}{2p_2^2} + \frac{1}{2} \right) m_2^2 - \left( \frac{m_2^2 - m_2^2}{2p_2^2} + \frac{1}{2} \right) m_2^2 = 0
\]

\[
= \frac{1}{4p_2^2} \left[ (p_2^2)^2 - 2p_2^2(m_2^2 + m_2^2) + (m_2^2 - m_2^2)^2 \right]
\]

\[
\implies p_2^2 = (m_2^2 + m_2^2)^2
\]
Box diagram

\[\begin{align*}
p_2 & \rightarrow q_2 \rightarrow p_3 \\
q_1 &= k_1 + p_1 \\
q_2 &= k_1 + p_1 + p_2 \\
q_3 &= -k_1 \\
q_4 &= k_1 + p_1 + p_2 - p_3
\end{align*}\]

\[
\sim \int \frac{1}{(2\pi)^d} \frac{d^4k_1}{(q_{11}^2 - m_1^2)(q_{21}^2 - m_2^2)(q_{31}^2 - m_3^2)(q_{41}^2 - m_4^2)}
\]

\[
= \int d\chi_1 \cdots d\chi_4 \int \frac{(4-1)!}{(2\pi)^d} \frac{\delta(1 - \chi_1 - \chi_2 - \chi_3 - \chi_4)}{[\chi_1(q_{11}^2 - m_1^2) + \chi_2(q_{21}^2 - m_2^2) + \chi_3(q_{31}^2 - m_3^2) + \chi_4(q_{41}^2 - m_4^2)]^4}
\]

\[\begin{align*}
\ldots &= k_1^2 - 2k_2 \left[ -\chi_1 - \chi_2 - \chi_4 \right] \left( \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \right) \\
&\quad + \left( \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \right) \left( \begin{array}{ccc} \chi_1 + \chi_2 + \chi_4 & \chi_2 + \chi_4 & -\chi_4 \\ \chi_2 + \chi_4 & \chi_2 + \chi_4 & -\chi_4 \\ -\chi_4 & -\chi_4 & \chi_4 \end{array} \right) \left( \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \right) - \sum_{i=1}^4 \chi_i m_i^2
\end{align*}\]

\[A = 1\]

\[B = \left( \begin{array}{ccc} -\chi_1 - \chi_2 - \chi_4 & -\chi_2 - \chi_4 & \chi_4 \\ -\chi_2 - \chi_4 & -\chi_2 - \chi_4 & \chi_4 \\ \chi_4 & \chi_4 & \chi_4 \end{array} \right)\]

\[C = \left( \begin{array}{ccc} \chi_1 + \chi_2 + \chi_4 & \chi_2 + \chi_4 & -\chi_4 \\ \chi_2 + \chi_4 & \chi_2 + \chi_4 & -\chi_4 \\ -\chi_4 & -\chi_4 & \chi_4 \end{array} \right)\]
After integrating over momenta,

\[ \text{(Box)} = \frac{3!}{(4\pi)^d/2} \gamma(4 - \frac{d}{2}) \frac{\Gamma(4 - \frac{d}{2})}{\Gamma(4)} \]

\[ \times \int d\chi_2 \ldots d\chi_n \delta(1 - \chi_1 - \ldots - \chi_n) \frac{1}{\left[ p_\mu \left( -B^\top \text{adj} A B + \Pi_{d+1} A \right) p^\mu - \chi_A m^2 \right]^{d/2}} \]