Feynman rule for $WWW$ vertex.

$$
\begin{align*}
\text{Use: } & W^+_\alpha |W^+_\alpha(p)\rangle \rightarrow g_{\mu\alpha} \\
& \partial_\mu W^+_\alpha(p) \rightarrow -ig_{\mu\alpha}
\end{align*}
$$

Feynman rule:

1. $i \times -ig\left(-ip\mu \ g_{\nu\alpha} - ip\nu \ g_{\mu\alpha}\right) \delta^\mu_{\beta} \delta^\nu_{\gamma} = -ig\left(p_{\beta} \ g_{\gamma\alpha} - p_{\gamma} \ g_{\beta\alpha}\right)$
2. $i \times -ig \times (-1) \left(-ik\mu \ g_{\nu\beta} - ik\nu \ g_{k\mu}\right) \delta^{\mu}_{\alpha} \delta^{\nu}_{\gamma} = ig\left(k_{\alpha} \ g_{\gamma\beta} - k_{\gamma} \ g_{\alpha\beta}\right)$
3. $i \times -ig \times (-i) \left(-ip\mu \ g_{\nu\gamma} - ip\nu \ g_{\mu\gamma}\right) \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} = -ig\left(q_{\alpha} \ g_{\nu\gamma} - q_{\nu} \ g_{\alpha\gamma}\right)$

Feynman rule $= -ig\left(g_{\alpha\beta}(k-p)^{\gamma} + g_{\beta\gamma}(q-k)^{\alpha} + g_{\alpha\gamma}(p-q)\right)$

Clean up:

$$
\begin{align*}
W^\pm W^\pm W^\pm &= -ig\left(g^{\mu\nu}(p^- - p^+)^{\rho} + g^{\nu\rho}(q^- - p^+)^{\mu} + g^{\rho\mu}(q^- - p^+)^{\nu}\right)
\end{align*}
$$

For $SM_2$ use $W_2^\pm = \cos\theta_W Z_\mu + \sin\theta_W A_{\mu}$

Agreed with Cheung & 6.
Feynman rule for $W^+W^-W^+W^-$ vertex

$$ \mathcal{L} = ... - \frac{i}{4} g^\mu_\nu g^{\mu_\sigma} \left( W_{\mu_\nu} W_{-\mu_\sigma} - W_{\mu_\sigma} W_{-\mu_\nu} \right) \left( W_{\rho_\mu} W_{-\rho_\nu} - W_{\rho_\nu} W_{-\rho_\mu} \right) \right) $$

antisymmetrize $\to \frac{1}{2}$, drop 2nd term $\to 2$

$$ = ... - \frac{i}{4} \left[ \frac{1}{2} \left( g^{\mu_\nu} g^{\mu_\sigma} - g^{\mu_\nu} g^{\mu_\sigma} \right) 2 W_{\mu_\nu} W_{-\mu_\sigma} W_{\rho_\mu} W_{-\rho_\nu} \right] $$

There are $2 \times 2 = 4$ possible contractions

Feynman rule $= i \times \frac{1}{2} g^2 \left( g^{\mu_\nu} g^{\mu_\sigma} - g^{\mu_\nu} g^{\mu_\sigma} \right) \times (q_{\mu_\nu} q_{\rho_\mu} + q_{\rho_\mu} q_{\mu_\nu}) (q_{\rho_\mu} q_{\nu_\sigma} + q_{\nu_\sigma} q_{\rho_\mu})$

$$ = \frac{i}{2} g^2 \left[ (g_{\mu_\nu} + g_{\mu_\sigma}) (g_{\rho_\mu} + g_{\rho_\nu}) - (\delta_\mu_\nu \delta_\rho_\sigma + \delta_\mu_\rho \delta_\nu_\sigma) (g_{\mu_\nu} q_{\rho_\mu} + q_{\rho_\mu} g_{\mu_\nu}) \right] $$

= \frac{i}{2} g^2 \left[ 4 g_{\mu_\nu} g_{\rho_\mu} - (g_{\rho_\mu} g_{\mu_\nu} + g_{\mu_\nu} g_{\rho_\mu} + g_{\rho_\mu} g_{\mu_\nu} + g_{\mu_\nu} g_{\rho_\mu}) \right] $$

= \frac{i}{2} g^2 \left[ 4 g_{\mu_\nu} g_{\rho_\mu} - 2 g_{\mu_\nu} g_{\rho_\mu} - 2 g_{\mu_\nu} g_{\rho_\mu} \right] $$

= -i g^2 \left( g_{\mu_\nu} g_{\rho_\mu} - 2 g_{\mu_\nu} g_{\rho_\mu} + g_{\mu_\nu} g_{\rho_\mu} \right) $$

Clean up

$$ = -i g^2 \left( g^{\mu_\nu} g^{\rho_\sigma} - 2 g^{\mu_\nu} g^{\rho_\sigma} + g^{\mu_\nu} g^{\rho_\sigma} \right) $$

like-choice

combinatorial factor $= \frac{1}{4}$
Feynman rule for $W^+ W^- W^3 W^3$

\[ L = - \frac{1}{2} g^{\mu \nu} g^{\rho \sigma} (W^+_{\mu} W^3_{\nu} - W^3_{\mu} W^+_{\nu}) (W^\rho W^3_{\sigma} - W^3 W^\rho) \]

\[ = - \frac{1}{2} \left( g^{\mu \nu} g^{\rho \sigma} - g^{\nu \rho} g^{\mu \sigma} \right) 2 W^+_{\mu} W^3_{\nu} 2 W^\rho W^3_{\sigma} \]

There are 2 possible contractions: $(W^a W^3 = A^a A^3$ or $Z^a Z^3$)
(There is 1 possible contraction if $W^a W^3 = A^3$, but extra $ZA$ counterterm)

Feynman rule:

\[ = -i g^2 \left( g^{\mu \nu} g^{\rho \sigma} - g^{\nu \rho} g^{\mu \sigma} \right) g_{\mu \nu} g_{\rho \sigma} (g_{\nu \rho} g_{\sigma \nu} + g_{\rho \sigma} g_{\mu \nu}) \]

\[ = -i g^2 (g_{\mu \nu} (g_{\rho \sigma} + g_{\sigma \nu}) - \delta^{\mu \rho} \delta^{\nu \sigma} (g_{\nu \rho} g_{\sigma \nu} + g_{\rho \sigma} g_{\mu \nu})) \]

\[ = i g^2 (g_{\mu \nu} g_{\rho \sigma} - 2 g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho}) \]

Clean up; $SU(2)$

\[ = i e^2 (g^{\mu \nu} g^{\rho \sigma} - 2 g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho}) \]

Combinatorial factor = $\frac{1}{2}$

For SM, $W^3 = \cos \theta_W Z^3_\mu + \sin \theta_W A^3_\mu \quad SU(2) \times U(1)$

\[ = i g^2 \cos^2 \theta_W (g^{\mu \nu} g^{\rho \sigma} - 2 g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho}) \]

\[ = i e g \cos \theta_W (g^{\mu \nu} g^{\rho \sigma} - 2 g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho}) \quad \text{Combinatorial factor} = 1 \]
Notice - there are no tree-level vertices of the type:

\[ ZZZ, \ \gamma ZZ, \ \gamma Z \text{ or } \gamma YY \]

This is obvious: \( \gamma \) couples to charged fields.

To understand this, write:

\[ Z = \cos \theta W + \sin \theta B \]

\[ \Rightarrow ZZZ = W^3 W^3 W^3 + W^3 W^3 B + W^3 B B + B B B \]

These are not present because \( B \) couples to hyper-charged fields.

\( (W^3 \text{ has no hypercharge, } Y = 0) \)

Finally: this is zero for elementary group-theoretic reasons:

Non-abelian gauge self couplings always involve the antisymmetric structure constants \( f^{abc} = e^{abc} \text{ for } SU(2) \).

\[ \Rightarrow L = \ldots g e^{abc} \gamma \ W^a \ W^b \ W^c \ \text{ will never contain } W^3 W^3 W^3. \]

n.b: similar arguments for absence of \( ZZZZ \) coupling.

\[ L = \ldots e^{abc} e^{cde} W^a \ W^b \ W^d \ W^e \ (ab) \text{ nor } (de) \text{ may be identical}. \]