Weak charged currents - leptons

Recall: \[ I_\ell = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \bar{\nu}_e \\ \bar{\nu}_\mu \\ \bar{\nu}_\tau \end{pmatrix} \quad \overline{c}_\ell = \begin{pmatrix} \bar{e} \\ \bar{\mu} \\ \bar{\tau} \end{pmatrix} \quad \text{reference:} \quad H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \]

Since there is no mixing in the lepton sector, consider just one generation:

\[ \mathcal{L} = I_\ell \, i \bar{\sigma}^\mu \not{D}_\mu \, I_\ell + \bar{e}_\ell \, i \bar{\sigma}^\mu \not{D}_\mu \, e - y_e \, H \, I_\ell \, \bar{e}_\ell + h.c. \]

Write out weak isospin products:

\[ \mathcal{L} = (\nu^\dagger \, e^\dagger) \, i \bar{\sigma}^\mu \left( \not{\partial}^\mu + \frac{i}{\sqrt{2}} \left( \not{W}^+ - \not{W}^- \right) \right) (\nu^\dagger) + \bar{e}_\ell \, i \bar{\sigma}^\mu \not{D}_\mu \, e - y_e \, (\phi^+ + \phi^0) \, (\nu^\dagger \, e^\dagger) \, \bar{e}_\ell + h.c. \]

\[ = \nu^\dagger \, i \bar{\sigma}^\mu \not{\partial}^\mu \nu + \bar{e}^\dagger \, i \bar{\sigma}^\mu \not{\partial}^\mu e - \frac{g}{\sqrt{2}} \, W^+_\mu \nu^\dagger \, i \bar{\sigma}^\mu e - \frac{g}{\sqrt{2}} \, W^-_\mu \bar{e}^\dagger \, i \bar{\sigma}^\mu \nu \]

\[ - y_e \, (\phi^+ + \phi^0) \, (\nu^\dagger \, e^\dagger) \, \bar{e}_\ell + h.c. \]

In four-component Dirac spinor notation:

\[ \mathcal{L} = \bar{\nu} (i \gamma^\mu) \nu + \bar{e} (i \gamma^\mu) e - \frac{g}{\sqrt{2}} \, W^+_\mu \bar{\nu} \gamma^\mu P_L \nu - \frac{g}{\sqrt{2}} \, W^-_\mu \bar{\nu} \gamma^\mu P_R \nu \]

\[ = \bar{\nu} (i \gamma^\mu) \nu + \bar{e} (i \gamma^\mu) e - \frac{g}{2 \sqrt{2}} \, W^+_\mu \bar{\nu} \gamma^\mu (1 - \gamma_5) e - \frac{g}{2 \sqrt{2}} \, W^-_\mu \bar{\nu} \gamma^\mu (1 - \gamma_5) \nu \]

\[ = \bar{\nu} (i \gamma^\mu) \nu + \bar{e} (i \gamma^\mu) e - \frac{g}{2 \sqrt{2}} \, W^+_\mu \bar{\nu} \gamma^\mu J^{-\mu} - \frac{g}{2 \sqrt{2}} \, W^-_\mu \bar{\nu} \gamma^\mu J^{+\mu} \]

\[ \text{lepton charged currents} \]
Polar decomposition - Mathematical aside.

Can decompose an arbitrary $n \times n$ complex matrix into a product of a Hermitian matrix $H$ and a unitary matrix $V$.

\[ M = HV \quad \text{(LEFT POLAR DECOMPOSITION)} \]

$M$ is complex and $H$ and $V$ are Hermitian.

\[ HH^\dagger = H \quad \text{(Hermitian)} \]
\[ (n^2 \text{ constraints}) \]
\[ V V^\dagger = I \quad \text{(Hermitian)} \]
\[ (n^2 \text{ constraints}) \]

$V$ is not unique. Provided $M$ is invertible, it is generally invertible.

\[ det(M) \neq 0 \quad \text{then} \quad det(M) = 0 \]

Unique? Yes, provided $M$ is invertible ($det(M) \neq 0$).

Construction:

Note $MM^\dagger = HH^\dagger = HH^\dagger$ is Hermitian. (Is Hermitian, hence, diagonalizable)

Diagonalize with unitary matrix $U$:

\[ UMM^\dagger U^\dagger = D^2 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \]

$\lambda_1, \lambda_2, \ldots$ positive (or zero)

Moreover, since $MM^\dagger = HH^\dagger$ (that is, it's the square of a Hermitian matrix), its eigenvalues are positive semi-definite (i.e. can have zero e-values).

So, $H$ is the principal square-root of $MM^\dagger$: $H = \sqrt{MM^\dagger}$

And therefore, the eigenvalues of $H$ are the positive square-root of those of $MM^\dagger$: $\lambda_{MM^\dagger} = \sqrt{\lambda_{MM^\dagger}}$.
\[ UHU^* = D = \begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \vdots \end{pmatrix} \quad \lambda_1, \lambda_2, \ldots \text{ positive (or zero)}. \]

Hence we can determine \( H \) by inverting the unitary transformation:

\[ H = U^* D U. \]

Having found \( H \), we can get \( V \) by inverting \( H \):

\[
\begin{array}{c}
\text{LEFT POLAR DECOMPOSITION} \\
M = H V \\
\Rightarrow [H^{-1} \cdot M = V]
\end{array}
\quad 
\begin{array}{c}
\text{RIGHT POLAR DECOMPOSITION} \\
M = V_R \cdot H \\
\Rightarrow [M \cdot H^{-1} = V_R]
\end{array}
\]

Obviously, this works only if \( H \) (and hence \( M \)) is invertible.

If not, the decomposition will not be unique as there will be an ambiguity as to what the phase of the zero eigenvalue is.

c.f. if \( z = 0 \) then \( z = e^{i\theta} \)

\[ \frac{0}{0} \text{ (ambiguous; } \theta \text{ can be anything).} \]

---

**Summary of Construction:** \( M = H V \)

1. Take product \( M M^* \)
2. Diagonalize \( U M M^* U^* = D^2 \) finding \( U \) and \( D^2 = (\lambda_1^2, \ldots) \)
3. Take square-root: \( D = \pm \sqrt{U M M^* U^*} \)
4. Transform back to get \( H = U^* \cdot D \cdot U = (U^* \sqrt{U M M^* U^*}) \cdot U \)
5. Invert to get \( V = H^{-1} \cdot V \) (or \( V = MH^{-1} \))
Diagonalization of arbitrary $n \times n$ complex matrix $M$:

Write $M = H \mathbf{V}$ (left polar decomposition; $H$ = Hermitian, $\mathbf{V}$ = Unitary)

Diagonalize $H$: $H = U^\dagger D U$

$M = U^\dagger D U \mathbf{V}$ (same $U$ from polar decomposition)

Combination $U \mathbf{V}$ is another unitary matrix $U'$

$M = U'^\dagger D U'$  \textbf{Singular Value Decomposition of $M$}

Thus, upon inverting $U^\dagger$ and $U'$

\[
UMU'^\dagger = \mathbf{D} = (\lambda_1, \lambda_2, \ldots)
\]

- $\lambda_1, \lambda_2, \ldots$ are the singular values of $M$
- Can always find $U$ & $U'$ so that $\lambda_1, \ldots$ are real

Therefore, in general, it takes two different unitary matrices $U$ & $U'$

to diagonalize an arbitrary $n \times n$ complex matrix:

- One Unitary matrix to render $M$ Hermitian.
- One Unitary matrix to diagonalize it.

\l A special unitary matrix would suffice. But leave the $U(1)$ phase in, and let physics decide if the phase is needed or not.

\textbf{n.b} The resulting diagonalized matrix has the same eigenvalues as those of $H$.