Lie Algebra

Definition:
Consists of vector space \( L \) over a field (\( \mathbb{R} \) or \( \mathbb{C} \), say) with a composition rule, called product, denoted \( \circ \), defined

\[
\circ : L \times L \rightarrow L
\]

If \( a_1, a_2, a_3 \in L \) are any three elements in \( L \), then the following defines the Lie algebra:

1. Closure \( a_1 \circ a_2 \in L \)
2. Linearity \( a_1 \circ (a_2 + a_3) = a_1 \circ a_2 + a_1 \circ a_3 \)
3. Antisymmetry \( a_1 \circ a_2 = -a_2 \circ a_1 \)
4. Jacobi Identity:
   \[
a_1 \circ (a_2 \circ a_3) + a_2 \circ (a_3 \circ a_1) + a_3 \circ (a_1 \circ a_2) = 0
   \]

For example, \( \{\sigma_z, \sigma_x, \sigma_y\} \) (Pauli sigma matrices) satisfies Lie Algebra properties

where \( \circ \) is commutator.
Graded Algebra

$\mathbb{Z}_2$ Graded:
Consists of vector space $L$ which is the direct sum of two subspaces, $L_0, L_1$:

$L = L_0 \oplus L_1$

and a product

$o : L \times L \to L$

with the following properties

Let $a_1, a_2 \in L_0$ and $b_1, b_2 \in L_1$

1. $a_1 \circ a_2 \in L_0$
2. $a_1 \circ b_1 \in L_1$
3. $b_2 \circ b_2 \in L_0$

sense of even & odd:

$a$ is even, $b$ is odd.

Example:
The set of all integers, $\mathbb{Z}$, under addition.

Here, (even integers) $\in L_0$
odd integers) $\in L_2$

Even + even = even
Even + odd = odd
Odd + odd = even

$\mathbb{Z}_n$ Graded:

Direct sum of $N+1$ vector spaces:

$L = L_0 \oplus L_1 \oplus L_2 \ldots \oplus L_N$

with product:

$o : L \times L \to L$

With property:

Let $a_k \in L_k$. then $a_j \circ a_k \in L_{j+k, \mod(N+1)}$

with this property, it's called grading.