Graded Lie Algebra

Is a graded algebra with the following additional properties:

For $\mathbb{Z}_2$-graded Lie algebra

1. Grading (obvious)
   
   For all $\alpha_i \in L_i$ \ $(i=0, 1)$,
   
   \[ x_i \circ x_j \in L_{i+j \mod 2} \]

2. Supersymmetrization (replaces antisymmetry)
   
   For all $x_i \in L_i$, $x_j \in L_j$, \ $(i,j = 0, 1)$,
   
   \[ x_i \circ x_j = -(-1)^{ij} x_j \circ x_i \]

3. Generalized Jacobi identities:
   
   For all $x_i \in L_i$, $x_j \in L_j$, $x_k \in L_k$, \ $(i,j,k = 0, 1)$:
   
   \[ (-1)^{ik} x_i \circ (x_j \circ x_k) + (-1)^{ij} x_j \circ (x_k \circ x_i) + (-1)^{jk} x_k \circ (x_i \circ x_j) = 0 \]

Although the set of integers form a $\mathbb{Z}_2$ graded algebra under addition, it does not form a $\mathbb{Z}_2$ graded Lie algebra because it does not obey supersymmetrization, i.e. $(1+3) \neq -(3+1)$
Graded Lie algebra of $SU(N)$

$L = L_0 \oplus L_1$, \[ L \times L \rightarrow L \]

Start with $L_0 = SU(N)$ algebra \( T^a \), \( a = \{1,2,..., N^2-1\} \)

\[ T^a \circ T^b = \{ T^a, T^b \} = i f^{abc} T^c \]

Now define $L_1$'s $Q_i$, \( i = \{1,2,..., n\} \) \( n = \) dimension of $L_1$

\[
\begin{align*}
T^a \circ Q_i & \rightarrow L_1 \\
Q_i \circ Q_j & \rightarrow L_0
\end{align*}
\]

Define $T^a \circ Q_i = - c_{ij} Q_j$ (minus sign for convenience)

The $c_{ij}$ are coefficients to be determined by the Jacobi identity

for $T^a, T^b, Q_i$:

\[
(-1)^0 T^a \circ (T^b \circ Q_i) + (-1)^0 T^b \circ (Q_i \circ T^a) + (-1)^0 Q_i \circ (T^a \circ T^b) = 0
\]

\[
T^a \circ (-c_{ij} Q_j) + T^b \circ c_{ij} Q_j + Q_i \circ i f^{abc} T^c = 0
\]

\[
c_{ij} C_{jk} Q_k + C_{ij} (-c_{jk} Q_k) + i f^{abc} C_{ik} Q_k = 0
\]

or, because $Q_k$ are independent,

\[
[c_{ij}, c_{ik}]_{jk} = i f^{abc} C_{ik} c_{jk}
\]

Hence, $c_{ij}$ matrices provide an $n \times n$ matrix representation of $L_0$.

Therefore, to construct a $\mathbb{Z}_2$ graded Lie algebra, we must pick a representation for $L_0$, and the number of $Q_i$'s matches the dimension of the representation.
Define: \( Q_i \circ Q_j = (Q_j \circ Q_i) = d_{ij}^a T^a \)

\( \text{symmetric in } i \neq j \).

Consider the Jacobi identity for \( T^a, Q_i, Q_j \):

\(-1\)^0 \( T^a \circ (Q_i \circ Q_j) + (-1)^0 Q_i \circ (Q_j \circ T^a) + (-1)^1 Q_j \circ (T^a \circ Q_i) = 0 \)

\[ T^a \circ d_{ij}^c T^c + Q_i \circ c_{jk}^a Q_k = Q_j \circ (-c_{jk}^a) Q_k = 0 \]

\[ d_{ij}^c \xi^{abc} T^b + c_{jk}^a d_{ik}^b T^b + c_{ic}^a d_{jk}^b T^b = 0 \]

or, because the \( T^b \) are independent,

\[ c_{ik}^a d_{kj}^b + c_{jk}^a d_{ki}^b = \xi^{abc} d_{ij}^c \quad \ast \]

Once \( c_{ij}^a \) are known (are representation matrices), can construct the \( d_{ij}^a \) by considering the above as a system of linear equations.

Example \( Z_2 \) graded \( SU(2) \), based on the fundamental representation:

- Choose representation: 2 (doublet \( \otimes \)) \( f_{abc} = \varepsilon_{abc} \)

Immediately, \( c^1 = \frac{1}{2}(1 \ 1), \quad c^2 = \frac{1}{2}(1, -1), \quad c^3 = \frac{1}{2}(1, 1) \)

Then let \( d^1 = (d_1^1, d_2^1, d_2^1), \quad d^2 = (d_1^2, d_2^2, d_2^2), \quad d^3 = (d_1^3, d_2^3, d_2^3) \)

and solve the system of equations in \( \ast \) [Mathematica]

Result: \( d^1 = N \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad d^2 = N \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad d^3 = N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

\( N \) sets the normalization of \( Q \).

Hence our graded algebra is: (there are five elements, \( T^1, T^2, T^3, Q_1, Q_2 \))

\( T^a \circ T^b = \varepsilon_{abc} T^c \)

\( T^a \circ Q_i = -c_{iak}^a Q_k \)

\( Q_i \circ Q_j = (d_{ij}^a) T^a \)

\( \text{OSp} (1/2) \quad \text{"one slash two"} \)

\( \text{now, } T^a, Q_i \text{'s can be any rep} \)
As a consistency check, the Jacobi identity for three $Q$s must be satisfied:

\[ (-1)^2 Q_i \circ (Q_j \circ Q_k) + (-1)^2 Q_j \circ (Q_k \circ Q_i) + (-1)^2 Q_k \circ (Q_i \circ Q_j) = 0 \]

\[ Q_i \circ d_{jk}^a T^a + Q_j \circ d_{ki}^a T^a + Q_k \circ d_{ij}^a T^a = 0 \]

\[ d_{jk}^a c_{ij}^b Q^b + d_{ki}^a c_{ji}^b Q^b + d_{ij}^a c_{k}^b Q^b = 0 \]

or, because $Q$ are indep,

\[ d_{jk}^a c_{ij}^b + d_{ki}^a c_{ji}^b + d_{ij}^a c_{k}^b = 0 \]

for $OSp(3|2)$ checked on Nairanain.
Determining $d_{ij}^*$

\[
c = \begin{bmatrix}
\frac{1}{2} & 0 & 1 \\
0 & -i & 0 \\
-1 & 0 & 1 \\
\end{bmatrix}, \\
d = \begin{bmatrix}
d_{11} & d_{1x} & d_{12} \\
d_{1x} & d_{21} & d_{2x} \\
d_{12} & d_{2x} & d_{22} \\
\end{bmatrix}, \\
\begin{bmatrix}
d_{31} & d_{3x} & d_{32} \\
\end{bmatrix};
\]

\[
\]

\[
\{d_{11} \rightarrow -d_{3x}, d_{21} \rightarrow i d_{3x}, d_{12} \rightarrow d_{3x}, d_{22} \rightarrow i d_{3x}, d_{2x} \rightarrow 0, d_{31} \rightarrow 0, d_{32} \rightarrow 0, d_{1x} \rightarrow 0\}
\]

### Consistency Check

\[
c = \begin{bmatrix}
\frac{1}{2} & 0 & 1 \\
0 & -i & 0 \\
-1 & 0 & 1 \\
\end{bmatrix}, \\
d = \begin{bmatrix}
-1 & 0 \\
0 & i & 0 \\
0 & 0 & 1 \\
\end{bmatrix};
\]

\[
\text{sum}[i_, j_, k_, l_] := \sum_{a=1}^{3} (d[a, j, k] \ast c[a, i, 1] + d[a, k, i] \ast c[a, j, 1] + d[a, i, j] \ast c[a, k, 1])
\]

\[
\text{Table}[
\text{sum}[i, j, k, l], (i, 1, 2), (j, 1, 2), (k, 1, 2), (l, 1, 2)]
\]

\[
\{(0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0)
\}
\]
More on $\mathbb{Z}_2$ graded Lie algebras

Generalized structure constants.

Let $L = L_0 \oplus L_1$ endowed with $\circ$ be our graded Lie Algebra.

$X_i \in L$ are elements of $L$.

$$X_i \circ X_j = c_{ij}^k X_k$$

$c_{ij}^k = \text{generalized structure constants}$.

$c_{ij}^k = -c_{ji}^k$ if $X_i, X_j \in L_0$ or $X_i \in L_0, X_j \in L_1$ even-even even-odd

$c_{ij}^k = +c_{ji}^k$ if $X_i, X_j \in L_1$ odd-odd

Jacobi identity. Let $T^a \in L_0$, $Q_i \in L_1$:

$$[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0$$

$$[T^a, [T^b, Q_i]] + [Q_i, [T^a, T^b]] + [T^b, [Q_i, T^a]] = 0$$

$$[T^a, \{Q_i, Q_j\}] - \{Q_j, [T^a, Q_i]\} + \{Q_i, [Q_j, T^a]\} = 0$$

[killing form]

$$[Q_i, \{Q_j, Q_k\}] + [Q_k, \{Q_i, Q_j\}] + [Q_j, \{Q_k, Q_i\}] = 0.$$