Problems 3

1. Consider a supersymmetric mechanical system of one superfield $X(t, \theta, \theta^*)$, interacting through the superpotential,

$$W(x) = C + \frac{1}{2}m\omega x^2 + \frac{1}{6}yx^3,$$

with $\omega, y > 0$. Write down the supersymmetric Lagrangian for the dynamical coordinates $x$ (describing a hypothetical particle), and $\psi$ and $\psi^*$ (abstract “Fermion particle”).

For simplicity, set $y = 0$ for now. Then repeat exercise with $y \neq 0$.

a) What is the potential $V(x)$ felt by the particle at $x$? make a sketch.

b) Falling back on the Pauli spin interpretation of $\psi$ and $\psi^*$, what is the magnetic induction (in the 3 direction) as a function of $x$ felt by the spin? make a sketch.

c) Small oscillation analysis

(i) Find the equilibrium stable points of the potential $V(x)$, and determine the small oscillation frequency about that point.

(ii) Suppose the spin of this particle is not aligned along the 3-direction. The spin will undergo Larmor precession. At the equilibrium points of the potential $V(x)$, determine the precession frequency for the spin about the 3 axes.

(iii) Compare to part (i), and admire the magic of supersymmetry! Qualitatively describe the simultaneous motion of the particle and its spin in the magnetic field.


a) Convince yourself that the Grassmann coordinate $\theta$ and $\theta^*$ carry dimensions $T/(LM^{1/2}) = (\text{energy})^{-1/2}$. Hint: use the supersymmetry algebra, and the unitary exponential form for a general supersymmetry transformation.

b) Write down the expansion for a general superfield $X(t, \theta, \theta^*)$ in terms of its component functions.

(i) Compute the derivative $\frac{\delta}{\delta \theta} X$, and determine the dimensions of the result.

(ii) Do the same for the integral $\int d\theta X$.

c) Does this make sense? Resolve paradoxes if any.
1. \[ W(x) = C + \frac{1}{2} m \omega^2 x^2 + \frac{1}{6} y x^3 \]

\[ S = \int dt \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2 m} (W')^2 + \frac{i}{4} y^2 \dot{\psi}^2 - \frac{1}{2 m} \left[ \psi^*, \psi^* \right] W^{\prime \prime}(x) \right) \]

\[ W'(x) = m \omega x + \frac{1}{2} y x^2 \Rightarrow (W')^2 = m^2 \omega^2 x^2 + m \omega y x^3 + \frac{1}{4} y^2 x^4 \]

\[ W''(x) = m \omega + y x \]

\[ \Rightarrow L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 - \frac{1}{2} m \omega y x^3 - \frac{3 y^2}{8 m} x^4 \]

\[ + \frac{i}{4} y^2 \dot{\psi}^2 - \omega \psi \dot{\psi} - \frac{1}{m} y \psi \dot{\psi} x \]

\( \gamma = \frac{\text{mechanical ratio}}{\gamma_m} \)

\[ \gamma_m = \frac{\omega}{\gamma} + \frac{y^2}{\gamma m} \]

b) \[ B_2(x) = \frac{1}{\gamma m} W''(x) \]

m=1.0, \( \omega=0.5 \), \( y=2.0 \), \( \gamma=1.0 \)
c) (ii) Equilibrium stable points:

Critical points:
\[
\frac{\partial V}{\partial x} = m\omega^2 x + \frac{3}{2} \omega y x + \frac{1}{2} \frac{y^2}{m} x^2 = 0
\]

solutions:
\[
x = \begin{cases} 
  \text{A}, & -\frac{2m\omega}{y}, \\
  \text{B}, & -\frac{m\omega^2}{y}
\end{cases}
\]

\[
\frac{\partial^2 V}{\partial x^2} = \begin{cases} 
  m\omega^2, & m\omega^2, \\
  \frac{-m\omega^2}{2}, & \frac{-m\omega^2}{2}
\end{cases}
\]

Stable points:

Oscillation frequencies:

\[
\begin{align*}
\chi_A &= 0 & \Rightarrow & \quad \omega_A = \omega \\
\chi_B &= -\frac{2m\omega}{y} & \Rightarrow & \quad \omega_B = \omega
\end{align*}
\]

(iii) Larmor precession:

\[
\bar{\omega}_L = -\gamma \bar{B}
\]

At point A:

\[
\bar{B}(x_A) = \frac{\omega}{\gamma} \Rightarrow \omega_L = -\omega
\]

At point B:

\[
\bar{B}(x_B) = \frac{\omega}{\gamma} + \frac{y}{m} \left( -\frac{2m\omega}{y} \right) = -\frac{\omega}{\gamma} \Rightarrow \omega_L = +\omega
\]

(iii) Oscillation frequency \( \omega_A, \omega_B \) equal to

Larmor precession frequency \( \omega_L \) \( \Rightarrow \)

\[
\begin{align*}
\text{Oscillation completes cycle in same time} & \\
\text{as precession completes cycle} & \Rightarrow \text{Synchronized}
\end{align*}
\]