Matrix representation for Grassmann generators

For \( N=1 \) (just one generator):

\[
\lambda = z_0 + z_1 \xi^1 \quad \text{rep}(\xi^1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

For \( N=2 \):

\[
\lambda = z_0 + z_1 \xi^1 + z_2 \xi^2 + z_{12} \xi^1 \xi^2
\]

\[
\text{rep}(\xi^1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{rep}(\xi^2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{rep}(\xi^1 \xi^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

For general \( N \),

need \( 2^N \times 2^N \) matrices.

Complex conjugation of supernumbers

The conjugation operator for complex numbers is defined:

\[
z = a + \mathbb{i}b \quad \Rightarrow \quad z^* = a - \mathbb{i}b
\]

For supernumbers, \( \lambda \in \Lambda_N \):

\[
\lambda = z_0 + z_i \xi^i + \frac{1}{2} z_{ij} \xi^i \xi^j + \ldots
\]

\[
\Rightarrow \quad \lambda^* = z_0^* + z_i^* (\xi^i)^* + \frac{1}{2} z_{ij}^* (\xi^i \xi^j)^* + \ldots
\]

we still need to define conjugation for Grassmann generators.

Convention:

\[
(\xi^i)^* = \xi^i \quad \text{0 reality condition}
\]

\[
(\xi^i \xi^j)^* = \xi^j \xi^i \quad \text{0 reverse order of multiplication}
\]

\[
(\xi^i \xi^j \xi^k)^* = \xi^k \xi^j \xi^i \quad \text{(no minus sign)}
\]

\[
\vdots
\]
Reality condition, together with reverse ordinary implies:

\[
(\xi \xi)^* = \xi^2
\]
\[
(\xi \xi^2 \xi)^* = \xi \xi^2 \xi = -\xi \xi^2 \xi
\]
\[
(\xi^2 \xi^2 \xi)^* = \xi^2 \xi^2 \xi = +\xi^2 \xi^2 \xi
\]

A supernumber \( \lambda \in \Lambda_N \) is said to be:

- purely real if \( \lambda^* = \lambda \) \( \implies \) happens if \( \varepsilon \)-even coefficients \( \in \text{Re} \) and \( \varepsilon \)-odd coefficients \( \in \text{Im} \).

- purely imaginary if \( \lambda^* = -\lambda \) \( \implies \) happens if \( \varepsilon \)-even coefficients \( \in \text{Im} \) and \( \varepsilon \)-odd coefficients \( \in \text{Re} \).

Note just because a supernumber is real doesn't mean it is real-valued (i.e., in \( \mathbb{R} \)) for example the QED Lagrangian: \( \mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - \frac{1}{2} m^2) \bar{\psi} \) is real but not real-valued.

In physics, it is useful to express supernumbers in terms of complex Grassmann generators:

Construction: (obviously) require \( \Lambda_N \) with \( N \) even.

Consider \( \Lambda_2 : \{ \xi^1, \xi^2 \} \).

Then \( \xi = \frac{1}{\sqrt{2}} \left( \xi^1 + i \xi^2 \right) \) \( \implies \)

\[
\xi^1 = \frac{1}{\sqrt{2}} \left( \xi + \xi^* \right)
\]
\[
\xi^2 = \frac{1}{\sqrt{2}} \left( \xi - \xi^* \right)
\]

A supernumber \( \lambda \in \Lambda_2 \) may be expressed:

\[
\lambda = z_0 + z_1 \xi^1 + z_2 \xi^2 + z_3 \xi^3 \xi^2
\]

\[
= z_0 + z_1 \frac{1}{\sqrt{2}} \left( \xi + \xi^* \right) + z_2 \frac{1}{\sqrt{2}} \left( \xi - \xi^* \right) + z_3 \frac{1}{2} \left( \xi + \xi^* \right) \left( \xi - \xi^* \right)
\]

organized by \( \xi, \xi^* \).
\[ \lambda = z_0 + \frac{1}{4Z} (z_1 - i z_2) \xi + \frac{1}{2Z} (z_1 + i z_2) \xi^* - \frac{i Z_2}{2} (z_5^* + z_5 \xi) \]

\[ = z_0 + z_1 \xi + z_2 \xi^* + i z_3 \xi \xi^* \]

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this is the Fockian
(Grassmann-odd) part

\( \xi \) used to ducabe Fermionic fields

- generator at each space & time point.
- alternatively, generator for each Focke mode.