Hamiltonian Mechanics - review

Lagrangian: \[ L = L(q^i, q^i, \ldots) \]

Conjugate momenta: \[ p_i = \frac{\partial L}{\partial \dot{q}^i} \]

Hamiltonian: \[ H = p_i \dot{q}^i - L \] (elim. \( \dot{q}^i \) in favor of \( p_i \)’s)

Hamilton’s equations of motion:
Vary the definition of the Hamiltonians,

\[ \delta H = p_i \delta q^i + \dot{q}^i \delta p_i - \frac{\partial}{\partial q^i} \left( \frac{\partial H}{\partial \dot{q}^i} \right) \delta q^i - \frac{\partial}{\partial \dot{q}^i} \left( \frac{\partial H}{\partial q^i} \right) \delta \dot{q}^i - \frac{\partial H}{\partial t} \delta t \]

\[ \approx \left( \dot{p}_i - \frac{\partial}{\partial q^i} \right) \delta q^i - p_i \dot{\delta q}^i + q^i \delta p_i - \frac{\partial H}{\partial t} \delta t \quad (1) \]

Vary the Hamiltonian as a general function: \( H(q, p, t) \)

\[ \delta H = \frac{\partial H}{\partial q^i} \delta q^i + \frac{\partial H}{\partial p_i} \delta p_i + \frac{\partial H}{\partial t} \delta t \quad (2) \]

Equate \( 1 = 2 \)

\[ 0 = \left( -\frac{\partial H}{\partial q^i} + \dot{p}_i \right) \delta q^i + \left( -\frac{\partial H}{\partial p_i} + q^i \right) \delta p_i + \left( -\frac{\partial H}{\partial t} - \frac{\partial H}{\partial t} \right) \delta t \]

Since the variations are independent, we arrive at the following equations of motion:

\[ -\dot{p}_i = \frac{\partial H}{\partial q^i} \quad \text{and} \quad \dot{q}^i = \frac{\partial H}{\partial p_i}; \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \]

\[ \frac{\partial H}{\partial q^i} + \dot{p}_i = 0 \quad \dot{q}^i - \frac{\partial H}{\partial p_i} = 0 \]
\[ S = \int_{t_1}^{t_2} dt \, L(q_i, \dot{q}_i) \quad i = 1, \ldots, N \]

Euler-Lagrange:
\[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \]

If \( L = L(q(t), \dot{q}(t)) \) (and indep of \( t \))
\[ \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \dot{q}_i - \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \dot{q}_j = 0 \]

or
\[ \ddot{q}_i = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \dot{q}_j \]

Inertial mass matrix.

Accelerations are uniquely determined by positions and velocities at a given time, \( t \) iff the matrix \( \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \) can be inverted to yield:
\[ \ddot{q}_i = \frac{\partial L}{\partial q_i} \left( \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right)^{-1} - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \left( \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right)^{-1} \dot{q}_j \]

This is important since forces should be functions of position \& velocity that are known at each time, \( t \). This is what yields predictable dynamics.

On the other hand, if inertial mass matrix \( \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \) is not invertible, then accelerations will not be known uniquely.

\( \overline{\text{To HAMILTONIAN}} \)

Perform a Legendre transformation: Define momenta: \( p_i = \frac{\partial L}{\partial \dot{q}_i} \)

However a passage over to the Hamiltonian requires us to eliminate \( \dot{q}_i \) in favor of \( p_i \). However, in order for this to work,

\[ \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \] needs to be invertible.
To HAMILTONIAN

\[
\begin{pmatrix}
q \\
p
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
A & B
\end{pmatrix}
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix}
\]

i.e. \( \dot{p}_i = A_{ij} \dot{q}_j + B_{ij} \dot{q}_j \)

Where for normal Lagrangians, yank change in velocities,

\( A = A(q) \) and \( B = B(q) \)

To eliminate velocities in favor of momenta, need inverse mapping:

\[
\begin{pmatrix}
q \\
\dot{q}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
-B^{-1}A & B^{-1}
\end{pmatrix}
\begin{pmatrix}
p \\
\dot{p}
\end{pmatrix}
\]

the inverse mapping exists if \( B^{-1} \) exists.

Since \( \ddot{p}_i = A(q)_{ij} \dot{q}_j + B(q)_{ij} \dot{q}_j \)

\[
\frac{\partial \ddot{p}_i}{\partial \dot{q}_j} = B(q)_{ij}
\]

but \( \ddot{p}_i = \frac{\partial L}{\partial \dot{q}_i} \)

so \[
\begin{vmatrix}
\frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}
\end{vmatrix} = B(q)_{ij}
\]

\[
\Rightarrow \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \quad \text{needs to be invertible.}
\]
Action in canonical form

\[ H = x p - L \]

or

If \( \Delta \) is invertible, I can write the action in Hamilton's form:

\[ L = x p - H \]

\[ S = \int dt \left( x p - H(x, p) \right) \]

\[ = \int dt \left( \frac{1}{2} (p \dot{x} - x \dot{p}) - H(x, p) \right) \]

The Euler-Lagrange equations of motion derived from this action are

Hamilton's equations of motion,

1. \[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \]
2. \[ \frac{\partial L}{\partial p} - \frac{d}{dt} \frac{\partial L}{\partial \dot{p}} = 0 \]

\[ -\frac{1}{2} \dot{p} - \frac{\partial H}{\partial x} - \frac{d}{dt} \left( \frac{1}{2} p^2 \right) = 0 \]

\[ \frac{1}{2} \dot{x} - \frac{\partial H}{\partial p} - \frac{d}{dt} \left( -\frac{1}{2} x^2 \right) = 0 \]

\[ \Rightarrow \dot{p} = \frac{-\partial H}{\partial x} \]

\[ \dot{x} = \frac{\partial H}{\partial p} \]