Suppose \( \frac{\partial^2 L}{\partial q_i \partial q_j} \) is not invertible.

In this case, not all of \( p_i = \frac{\partial L}{\partial \dot{q}_i} \) may be used to define conjugate momenta.

Some of these equations will result in relations among the various momenta & positions:

\[ \delta m (q, p) = 0 \quad \text{These are the constant equations.} \]
\[ m = \{1, 2, ..., M\}, \quad M \text{ independent constants} \]

These equations define a surface (sub-manifold) of dimension \( 2N - M \)

in the \( 2N \) dimensional phase space. \( \Sigma \) is PRIMARY CONSTRAINT SURFACE  \( \Sigma_{\text{constr.}} \).

where \( \delta m (q, p) \) is satisfied.

The points which satisfy \( \delta m (q, p) = 0 \).

\[ \Sigma_{\text{constr.}} \text{ satisfy } \delta m (q, p) = 0. \]

2\( N \) dimensional phase space. [Whole space would be used if \( \frac{\partial^2 L}{\partial q_i \partial q_j} \) were invertible].

Rank of \( \frac{\partial^2 L}{\partial q_i \partial q_j} \) tells us how many independent degrees of freedom there really are.

If rank is less than \( N \) (the size of \( \frac{\partial^2 L}{\partial q_i \partial q_j} \)), then its determinant vanishes \( \Rightarrow \) not invertible.

\[
\text{Rank} \left( \frac{\partial^2 L}{\partial q_i \partial q_j} \right) = N - M \equiv N_{\text{true}}
\]

coordinate constraints \( \rightarrow \) true dynamical degrees of freedom.
Given a point \((q_i, p_i)\) satisfying \(\mathbf{M}(q, p) = 0\), there will be many points \((q_i, \dot{q}_i)\) that solve \(\pi_i = \frac{\partial L}{\partial \dot{q}_i}\).

The definition \(\pi_i = \frac{\partial L}{\partial \dot{q}_i}\) defines a map from the smaller \(\Sigma(q, p)\) manifold to the larger \((q, \dot{q})\) canonical phase space.

That there are infinitely many points of phase space corresponding to a single point on \(\Sigma(q, p)\) is the direct result of \(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\) being non-invertible.
Suppose \( \frac{\partial^2 \mathcal{L}}{\partial q_i \partial q_j} \) is not invertible:

In this case, there will be relations among the various momenta and positions coming through the definition of \( p_i = \frac{\partial \mathcal{L}}{\partial q_i} \):

\[
\dot{q}_m(q, p) = 0 \quad \text{in general, there may be many, but let there be just } M \text{ independent constraints here.}
\]

These equations define a surface (submanifold) in \( \Sigma \) PRIMARY CONSTRAINT SURFACE, \( \Sigma \) cons th pase space.

The rank of \( \frac{\partial^2 \mathcal{L}}{\partial q_i \partial q_j} \) tells us how many independent DOF there really are. If the rank is less than \( N \) (the size of \( \frac{\partial^2 \mathcal{L}}{\partial q_j^2} \)) then its determinant vanishes \( \Rightarrow \) non-invertible.

The rank of \( \frac{\partial^2 \mathcal{L}}{\partial q_i \partial q_j} \) is equal to \( N - M = N - \text{true} \) constraints \( \Rightarrow \) True dynamical degrees of freedom.

Since phase space is \( 2N \)-dimensional (for the \( q \) & \( p \)), the PRIMARY constraint surface will be \( 2N - M \) dimensional.

Given a point \( (q, p) \) satisfying \( \dot{q}_m(q, p) = 0 \), there will be many points \( (q', p') \) that solve \( p = \frac{\partial \mathcal{L}}{\partial q} \).

The definition \( p_i = \frac{\partial \mathcal{L}}{\partial q_i} \) defines a map from the smaller \( (q, p) \) manifold to the larger \( (q, q') \) manifold. The inverse images of a point of on \( \dot{q}_m(q, p) \) form an \( M' \)-dimensional manifold.

\( \Sigma \) cons, \( (2N-M')\text{-dimensional} \)

\( \text{configuration space} \quad q \)

\( \text{symplectic phase space} \quad q \)

\( (\mu\text{-space}) \text{ used in Boltzmann Eqn.} \)
In Lagrangian mechanics, if the system has holonomic constraints, then one generally adds Lagrange multipliers to the Lagrangian to account for them.

These extra terms vanish when constraints are satisfied.

→ More to Hamiltonian mechanics, and we have a constraint in phase space.

Primary constraint: Definition of momenta not invertible ⇒ they instead give
a constraint in phase space.

Note: The Hamiltonian may always be written in canonical form; even if
some of the momenta are not invertible ⇒ give constraints $\mathfrak{h}(q,p) = 0$.

\[ H = H(p,q) \quad \text{"Canonical Hamiltonian"} \]

Hamiltonian is no longer unique, since we may add to it $N$ primary constraints which
vanish on-shell.

The most general Hamiltonian is: (à la Dirac)

\[ H_{gen} = H + \sum_{m} \phi_m(q,p) \partial \mathfrak{h}(q,p) \approx H \]

RECALL, starting from $H = \dot{q}p - L$, we vary $p$ and $q$ independently to set $\partial q = 0$

\[ \frac{dH}{dt} = \dot{q} \delta p + \dot{p} \delta q - \frac{\partial L}{\partial q} \delta q - \frac{\partial L}{\partial p} \delta p - \frac{\partial L}{\partial \dot{q}} \dot{\delta q} \]

and also,

\[ \dot{p} \delta q + \dot{q} \delta p \approx -\dot{\delta q} - \dot{\delta p} \]

$\delta p$ is not an independent.

Variation, but a linear comb of $\delta q$ & $\delta p$. Always at the
form (cnmt) $\delta q = \delta p = 0$. It is
always a function of $p \& q$, even
though some details of $p \& q$ are
non-invertible.

\[ \left( -\frac{\partial H}{\partial q} - \dot{q} \right) \delta q + \left( -\frac{\partial H}{\partial p} + \dot{p} \right) \delta p = 0 \]

This is Hamilton's eqn of motion.

With constraints now, $\delta p$ and $\delta q$ are not independent differentials (variations).

They are related via the constraints which must vanish on-shell.

(i.e. the variations must be tangential to the surface of constraint)
The solution to \[ \sum_i A_i(p, q) \delta q_i + \sum_i B_i(p, q) \delta p_i = 0 \] is

1. \[ A_i(p, q) = \sum_j u_j(q, i) \frac{\partial \phi_j}{\partial q_i} \] (recall \( t_i \) is zero without constraint)

2. \[ B_i(p, q) = \sum_j u_j(q, i) \frac{\partial \phi_j}{\partial p_i} \] Here, \( u_j(q, i) \) are functions of positions and velocities.

The result is (for the Hamiltonian system)

1. \[ \frac{\partial H}{\partial \dot{q}_i} - \dot{p}_i \approx \sum_j u_j(q, i) \frac{\partial \phi_j}{\partial q_i} \] (Equations of motion, obviously on-shell condition)

2. \[ \frac{\partial H}{\partial \dot{p}_i} + \dot{q}_i \approx \sum_j u_j(q, i) \frac{\partial \phi_j}{\partial \dot{q}_i} \]

3. \[ \phi(q, p) \approx 0 \]

Notice \( \dot{q}_i = \frac{\partial H}{\partial p_i} + \sum_j u_j(q, i) \frac{\partial \phi_j}{\partial q_i} \)

Now we know \( \dot{q}_i \) if \( q_i \)'s and \( p_i \)'s are known if we know \( u_j(q, i) \)

Different points here correspond to different \( u_j(q, i) \). The \( u_j(q, i) \) parameterize the surface.
Recall, \( \{ \dot{f}, H \}_{PB} = \sum_i \left( \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial \dot{q}_i} - \frac{\partial f}{\partial \dot{q}_i} \frac{\partial H}{\partial q_i} \right) \)

or \( \sum_i \left( \frac{\partial f}{\partial q_i} (\dot{q}_i - \sum_j u_j \frac{\partial f_j}{\partial \dot{q}_j}) - \frac{\partial f}{\partial \dot{q}_i} (\dot{p}_i - \sum_j u_j \frac{\partial f_j}{\partial p_j}) \right) = \frac{df}{dt}(q_1, ..., p_j, ...) \)

gives the time evolution of the function (as long as the equations of motion are satisfied).

With the generalized Hamiltonian, we have:

\[ \{ f, H_{gen} \}_{PB} = \sum_i \left( \frac{\partial f}{\partial q_i} \frac{\partial H_{gen}}{\partial \dot{q}_i} - \frac{\partial f}{\partial \dot{q}_i} \frac{\partial H_{gen}}{\partial q_i} \right) \]

\[ = \sum_i \left( \frac{\partial f}{\partial q_i} (\dot{q}_i - \sum_j u_j \frac{\partial f_j}{\partial \dot{q}_j}) - \frac{\partial f}{\partial \dot{q}_i} (\dot{p}_i - \sum_j u_j \frac{\partial f_j}{\partial p_j}) \right) \]

\[ = \frac{df}{dt}(q_1, ..., p_j, ...) + \sum_i \sum_j u_j \left( \frac{\partial f}{\partial q_i} \frac{\partial f_j}{\partial \dot{q}_i} + \frac{\partial f}{\partial \dot{q}_i} \frac{\partial f_j}{\partial p_j} \right) \]

\[ = \frac{df}{dt}(q_1, ..., p_j, ...) + \sum_j u_j \{ f, f_j \}_{PB} \]

So, \( \frac{df}{dt}(q_1, ..., p_j, ...) \approx \{ f, H_{gen} \}_{PB} + \sum_j u_j \{ f, f_j \}_{PB} \).

Now, important that the constraint conditions are time-independent when

tors are satisfied. Why? Because otherwise the evolution of the

system would knock it out of the surface of constant. Require \( \dot{p}_j \approx 0 \)

when tors satisfied

\[ \{ \dot{f}, H_{gen} \}_{PB} \approx \sum_i \left( \frac{\partial f}{\partial q_i} \frac{\partial H_{gen}}{\partial \dot{q}_i} - \frac{\partial f}{\partial \dot{q}_i} \frac{\partial H_{gen}}{\partial q_i} \right) \]

\[ = \frac{df}{dt}(q_1, ..., p_j, ...) - \sum_k u_k \{ f, \phi_k \}_{PB} \]

\[ \text{demand} = 0 \]

So: \( \{ \phi_j, H_{gen} \}_{PB} \approx -\sum_k u_k \{ \phi_j, \phi_k \}_{PB} \) Consistency condition
Four possibilities with constancy conditions:

- an inconsistent eqn (e.g., $x = 0$) arises from $L = q$.
- an equation that is trivially true.
- An equation placing new constraints on coordinates/moments, but independent of the $q_i$.
- An equation that helps fix the $q_i$.

2. A constraint derived in this manner is called a secondary constraint. This must be added to the Hamiltonian, and constancy checked again.

(note: the division between primary & secondary constraints is arbitrary; could have just stated with both constraints).

If in the end, all the $q_i$'s are not fixed, then there is/are redundant (unphysical) degrees of freedom.

Otherwise, substitute $q_i$'s into $H_{ext} \Rightarrow$ given Hamiltonian whose equation of motion match $E=1$, eq. of mot.

The Dirac brackets are to be "upgraded" to commutation relations.