Example: with 1st class constraint

Notion of a particle confined to the surface of a 2-sphere.
Let us describe motion of particle with four coordinates. \( q_i = (x, y, z, \lambda) \)

\[
L = \frac{1}{2} m (x^2 + y^2 + z^2) - \frac{1}{2} \lambda (x^2 + y^2 + z^2 - R^2) \quad R = \text{const., dist.}
\]

Kinetic term \( (\lambda \text{ absent}) \)

Constraints system to surface of sphere.

We shall see using Hamiltonian mechanics that \( \lambda \) has no independent dynamics. Its Euler-Lagrange equation of motion gives the constraint:

\[
\frac{\partial L}{\partial \dot{\lambda}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\lambda}} = -\frac{1}{2} (x^2 + y^2 + z^2 - R^2) = 0.
\]

Notice the (inertial) mass matrix is \( \frac{\partial^2 L}{\partial q_i \partial q_j} = \begin{pmatrix} m & m \\ m & 0 \end{pmatrix} \) not invariant.

To Hamiltonian mechanics,

\[
p_x = \frac{\partial L}{\partial \dot{x}} = mx \Rightarrow \dot{x} = \frac{p_x}{m}
\]

\[
p_y = \frac{\partial L}{\partial \dot{y}} = my \Rightarrow \dot{y} = \frac{p_y}{m}
\]

\[
p_z = \frac{\partial L}{\partial \dot{z}} = m \dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}
\]

\[
p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0 \quad \rightarrow \quad p_\lambda = 0 \quad \text{Primary constraint}
\]

Canonical Hamiltonian "Naive Hamiltonian."

\[
H_{\text{can}} = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} + p_\lambda \dot{\lambda} = L
\]

\[
= \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + \frac{1}{2} \lambda (x^2 + y^2 + z^2 - R^2)
\]
Then, the extended Hamiltonian is:

\[ H_{\text{ext}} = H_{\text{can}} + u_1 \phi_1(p, q) \]

\[ = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + \frac{1}{2} \lambda (x^2 + y^2 + z^2 - R^2) + u_1 p_\lambda \]

Now require primary constraint to be time-independent on shell.

\[ \phi_1(p, q) = \{ \phi_1(p, q), \ H_{\text{ext}} \} \approx 0 \]

\[ = \sum_i \left( \frac{\partial \phi_1}{\partial q_i} \frac{\partial H_{\text{ext}}}{\partial p_i} - \frac{\partial \phi_1}{\partial p_i} \frac{\partial H_{\text{ext}}}{\partial q_i} \right) \]

\[ i = \{ \lambda \} \text{ term contributes} \]

\[ = -\frac{\partial H_{\text{ext}}}{\partial \lambda} = -\frac{1}{2} (x^2 + y^2 + z^2 - R^2) \approx 0 \quad \text{Secondary constraint} \]

This does not serve to fix \( u_1(p, q) \). Instead we have a new constraint — one which keeps the particle confined to surface of sphere.

Let \( \frac{1}{2} (x^2 + y^2 + z^2 - R^2) = \phi_2(p, q) \). Add to Hamiltonian.

\[ H_{\text{ext}} \rightarrow H_{\text{ext}} + u_2(p, q) \phi_2(p, q) \]

\[ = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + \frac{1}{2} (\lambda + u_2) (x^2 + y^2 + z^2 - R^2) + u_1 p_\lambda \]

and require time-independence of new constraint.

\[ \phi_2(p, q) = \{ \phi_2(p, q), \ H_{\text{ext}} \} \approx 0 \]

\[ = \sum_i \left( \frac{\partial \phi_2}{\partial q_i} \frac{\partial H_{\text{ext}}}{\partial p_i} - \frac{\partial \phi_2}{\partial p_i} \frac{\partial H_{\text{ext}}}{\partial q_i} \right) \]

\[ i = \{ x, y, z \} \]

Only first term contributes with \( i = \{ x, y, z \} \).
\[ \{ \dot{\phi}_3(p, q), \text{ext} \} = (x \cdot \frac{1}{m} p_x + y \cdot \frac{1}{m} p_y + z \cdot \frac{1}{m} p_z) \]

\[ = \frac{1}{m} (x p_x + y p_y + z p_z) \approx 0 \quad \text{(Hermitian constraint)} \]

We have yet another constraint — this time it is \( \mathbf{T} \cdot \mathbf{p} \approx 0 \); the momentum must be tangential to the surface of a sphere (makes sense).

Let \( \dot{\phi}_3(p, q) = x p_x + y p_y + z p_z \) and add to Hamiltonian,

\[ \text{ext} \rightarrow \text{ext} + u_3 \dot{\phi}_3(p, q) \]

\[ = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + u_3 (x p_x + y p_y + z p_z) \]

\[ + \frac{1}{2} (\lambda + u_2) (x^2 + y^2 + z^2 - R^2) + u_1 p_\lambda \]

\[ = \frac{1}{2m} \mathbf{p}^2 + u_3 \mathbf{T} \cdot \mathbf{p} + \frac{1}{2} (\lambda + u_2) (\mathbf{p}^2 - R^2) + u_1 p_\lambda \]

\[ \mathbf{T} = (x, y, z) \quad \mathbf{p} = (p_x, p_y, p_z) \]

and require new constraint to be time independent,

\[ \dot{\phi}_3(p, q) = \{ \phi_3(p, q), \text{ext} \} = \sum_i \left( \frac{\partial \phi_3}{\partial p_i} \frac{\partial \text{ext}}{\partial q_i} - \frac{\partial \phi_3}{\partial q_i} \frac{\partial \text{ext}}{\partial p_i} \right) \]

\[ = \mathbf{p} \cdot \left( \frac{1}{m} \mathbf{p} + u_3 \mathbf{T} \right) - \mathbf{T} \cdot \left( u_3 \mathbf{p} + (\lambda + u_2) \mathbf{p} \right) \]

\[ = \frac{1}{m} \mathbf{p}^2 - (\lambda + u_2) \mathbf{p}^2 \approx 0 \]

This equation is not a new constraint; it serves to fix \( u_2 \):

\[ u_2 = \frac{1}{m} \frac{\mathbf{p}^2}{r^2} - \lambda \]

do not use on-shell condition to replace \( r^2 = R^2 \) yet.
Fixing the $u_3$. Go back and require that $\phi_1$, $\phi_2$ are time independent with the new extended Hamiltonian.

$$\dot{\phi}_1(p, q) = \{\phi_1, H_{ext}\} = 0$$

$$= \sum_i \left( \frac{\partial \phi_1}{\partial q_i} \frac{\partial H_{ext}}{\partial p_i} - \frac{\partial \phi_1}{\partial p_i} \frac{\partial H_{ext}}{\partial q_i} \right)$$

$$= - (p^2 - R^2) \approx 0$$

Does not help – cannot determine $u_3$ (will see later why).

$$\dot{\phi}_2(p, q) = \{\phi_2, H_{ext}\} = 0$$

$$= \sum_i \left( \frac{\partial \phi_2}{\partial q_i} \frac{\partial H_{ext}}{\partial p_i} - \frac{\partial \phi_2}{\partial p_i} \frac{\partial H_{ext}}{\partial q_i} \right)$$

$$= \dot{p}_1 \left( \frac{1}{m} \dot{p} + u_3 \right) = \frac{1}{m} \dot{p}_1 \dot{p} + u_3 \dot{p}$$

This helps to determine $u_3 = - \frac{\dot{p}_1 \dot{p}}{m \dot{p}^2}$.

Thus, the final form of the Hamiltonian is:

$$H_{ext} = \frac{1}{2m} \dot{p}^2 - \frac{\dot{p}_1 \dot{p}}{m \dot{p}^2} \dot{p}_1 \dot{p} + \frac{1}{2} \left( \lambda + \frac{1}{m} \dot{p}^2 - \lambda \right) (p^2 - R^2) + u_3 \dot{p}_1$$

$$= \frac{1}{2m} \dot{p}^2 - \frac{(\dot{p}_1 \dot{p})^2}{m \dot{p}^2} + \frac{1}{2m} \dot{p}^2 \left( 1 - \frac{R^2}{\dot{p}^2} \right) + u_3 \dot{p}_1$$

$$H_{ext} = \frac{1}{2m} \left( 2 - \frac{R^2}{\dot{p}^2} \right) \dot{p}^2 - \frac{(\dot{p}_1 \dot{p})^2}{m \dot{p}^2} + u_3 \dot{p}_1$$

Where the constraints are:

$$\phi_1(p, r) = \dot{p}_1$$

$$\phi_2(p, r) = \frac{1}{2} (p^2 - R^2)$$

particle remains on sphere

$$\phi_3(p, r) = \dot{p} \dot{p}$$

momentum tangent to surface of sphere.
Poisson brackets:

\[ \{ \phi_2, \phi_2 \} = 0 \]
\[ \{ \phi_4, \phi_3 \} = 0 \]
\[ \{ \phi_2, \phi_3 \} = \sum_i \left( \frac{\partial \phi_2}{\partial q_i} \frac{\partial \phi_3}{\partial \dot{q}_i} - \frac{\partial \phi_2}{\partial \dot{q}_i} \frac{\partial \phi_3}{\partial q_i} \right) = r/2 \]

Involve different degrees of freedom

Thus, \( \phi_2 \) and \( \phi_3 \) are second class constraints.

But, \( \phi_4 \) is a first class constraint, and signals the presence of a gauge redundancy in the system.

Next, convert the first class constraint \( \phi_4 = p_\lambda = 0 \) into a second class constraint by imposing a gauge-fixing condition:

\[ \phi_4 = \lambda - \alpha = 0 \]

arbitrary number
(can choose \( \alpha = 0 \) for simplicity).
Fixing the gauge:

Convert the first class constraint $\phi_1 = p_\lambda = 0$ into a second class constraint by imposing a gauge-fixing condition:

$$\phi_q = \lambda - \alpha = 0$$

arbitrary number

(can choose $\alpha = 0$ for simplicity)

Check: Bracket involving $\phi_1$ no longer vanishes:

$$\{\phi_1, \phi_q\} = \sum_i \left( \frac{\partial \phi_1}{\partial q_i} \frac{\partial \phi_q}{\partial p_i} - \frac{\partial \phi_1}{\partial p_i} \frac{\partial \phi_q}{\partial q_i} \right) \phi_1(x, y, z, \lambda)$$

$$= 0' - (1)(1) = -1 \quad \text{GOOD!} \quad \phi_1 \text{ has become second class.}$$

New consistency condition:

$$\dot{\phi}_q = \{\phi_q, H\} = \sum_i \left( \frac{\partial \phi_q}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \phi_q}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0$$

$$= 1 \frac{\partial H}{\partial p_\lambda} - 0 = 0$$

$$= u_\lambda = 0 \quad \Rightarrow \text{Fix } u_\lambda (1)$$

So the (gauge-fixed) extended Hamiltonian is (set $u_\lambda = 0$):

$$H_{\text{ext}} = \frac{1}{2m} \left( 2 - \frac{\xi^2}{r^2} \right) \dot{r}^2 - \frac{(\dot{r}^2 + \dot{\phi}^2)}{m r^2}$$

Proceed to compute all other Poisson brackets:

$$\{\phi_2, \phi_q\} = \sum_i \left( \frac{\partial \phi_2}{\partial q_i} \frac{\partial \phi_q}{\partial p_i} - \frac{\partial \phi_2}{\partial p_i} \frac{\partial \phi_q}{\partial q_i} \right) = 0$$

$$\{\phi_3, \phi_q\} = \sum_{i} \left( \frac{\partial \phi_3}{\partial q_i} \frac{\partial \phi_q}{\partial p_i} - \frac{\partial \phi_3}{\partial p_i} \frac{\partial \phi_q}{\partial q_i} \right) = 0$$
so Poisson bracket matrix is:

\[ M_{ab} = \left\{ \varphi_a, \varphi_b \right\}_{PB} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} \]

inverse:

\[ (M^{-1})_{ab} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} \]

\[ \varphi_4 = P_\lambda \]
\[ \varphi_2 = \frac{1}{2} (r^2 - R^2) \]
\[ \varphi_3 = \vec{r} \cdot \vec{p} \]
\[ \varphi_4 = \lambda \]

Dirac Bracket:

\[ \{ f, g \}_{DB} = \{ f, g \}_{PB} - \sum_{a, b} \{ f, \varphi_a \}_{PB} (M^{-1})_{ab} \{ \varphi_b, g \}_{PB} \]

For quantization, need all Dirac brackets involving \( x_i, p_i, \lambda, P_\lambda \):

\[ \{ q_i, \varphi_a \}_{PB} = \frac{\partial \varphi_a}{\partial q_i} \Rightarrow \{ x_i, \varphi_3 \} = 0 \]
\[ \{ x_i, \varphi_2 \} = 0 \]
\[ \{ x_i, \varphi_4 \} = x_i \]
\[ \{ \lambda, \varphi_4 \} = 0 \]

\[ \{ p_i, \varphi_a \}_{PB} = -\frac{\partial \varphi_a}{\partial q_i} \Rightarrow \{ p_i, \varphi_4 \} = 0 \]
\[ \{ p_i, \varphi_3 \} = -p_i \]
\[ \{ p_i, \varphi_3 \} = 0 \]
\[ \{ p_\lambda, \varphi_4 \} = -1 \]