Identities

Symmetry properties of $V(\Phi)$ impose constraints on its expansion coefficients, $M^2_{ij}$, $c_{ij}$ & $C_{ijkl}$:

If $V(\Phi)$ is invariant under some group (not necessarily simple), then:

$$V'(\Phi') = V(\Phi) \quad \text{where} \quad \Phi_i' = (e^{i\alpha^a gT^a})_{ij} \Phi_j \Rightarrow (1 + \alpha^a gT^a)_{ij} \Phi_j \Phi_j \approx (1 + \alpha^a gT^a)_{ij} \Phi_j \Phi_j$$

Then

$$V'(\Phi') = V'((1 + \alpha^a gT^a)\Phi)$$

$$\Rightarrow \left( gT^a \Phi \right)_i \frac{\partial V}{\partial \Phi_i} = 0$$

since eqn true for any $\alpha^a$.

Now differentiate this $\uparrow$ with respect to $\Phi_j$.

$$\left( gT^a \Phi \right)_i \frac{\partial V}{\partial \Phi_i} + (gT^a \Phi)_i \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} = 0$$

or

$$\Rightarrow \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} = M^2_{ij}$$

Note: $M^2_{ij}(\Phi)$ is the shifted mass matrix when all scalars are shifted.

When this identity is evaluated at the tree level minimum, $\langle \Phi \rangle = \Phi_0$, we have

$$\frac{\partial V}{\partial \Phi_j} \langle \Phi \rangle = 0 \Rightarrow \left( gT^a \langle \Phi \rangle \right)_i M^2_{ij} = 0$$

or

$$(m^2_A)_{ki} M^2_{ij} = 0$$

where $M^2_{ij}$ is the tree-level mass matrix for scalar fields.

This means at the tree-level minimum,

$$[m^2_A, M^2] = 0 \quad \text{and} \quad m^2_A \& M^2 \text{ are simultaneously diagonalizable. (very important)}$$
Note that $m^2(\phi)^{ab}_{\phi} = (gT^a\phi)_i (gT^b\phi)_i$ and $m^2(\phi)_{ij} = (gT^a\phi)_i (gT^a\phi)_j$ have the same non-zero eigenvalues — even as functions of $\phi$.

Proof: Let $\omega^b$ be an e-vector of $m^2(\phi)^{ab}_{\phi}$ with e-value $\lambda$.

$$m^2(\phi)^{ab}_{\phi} \omega^b = (gT^a\phi)_i (gT^b\phi)_i \omega^b = \lambda \omega^a$$

Now multiply by $(gT^a\phi)_j$ and sum over $a$.

$$\frac{(gT^a\phi)_j (gT^a\phi)_i (gT^b\phi)_i \omega^b = \lambda (gT^a\phi)_j \omega^a}{m^2(\phi)_{ij} \omega^a} \quad \text{Define: } N \omega_i \quad \text{and } N \omega_j$$

$$m^2(\phi)_{ij} N \omega_i = \lambda N \omega_j$$

or $$m^2(\phi)_{ij} N \omega_j = \lambda N \omega_i \tag*{\blacksquare}$$

To get the normalization for $\omega_j$ (the e-vector of $m^2(\phi)_{ij}$), suppose $\omega^a$ is normalized: $\omega^a \omega^a = 1$

Then $$\omega_i \omega_i = \frac{1}{N^2} (gT^a\phi)_i \omega^a (gT^b\phi)_i \omega^b$$

$$= \frac{1}{N^2} (gT^a\phi)_i (gT^b\phi)_i \omega^b \omega^a$$

$$m^2(\phi)^{ab}_{\phi} \omega^b = \lambda \omega^a$$

$$= \frac{1}{N^2} \lambda \frac{\omega^a \omega^a}{1} = \frac{1}{N^2} \lambda \cdot 1 = 1$$

Then $$\Rightarrow N = \frac{1}{\sqrt{\lambda}}$$

So, if $\omega^b$ is an e-vector of $m^2(\phi)^{ab}_{\phi}$ with e-value $\lambda$, then $\omega_j = \frac{1}{N\lambda} (gT^a\phi)_j \omega^a$ is an e-vector of $m^2(\phi)_{ij}$ with the same e-value.

Note: Nothing can be said about e-vectors with zero e-values. In fact, $(m^2)^{ab}_{\phi}$ and $(m^2)_{ij}$ may have differing number of zero e-values.